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Reflections on the Foundations of Mathematics

Univalent Foundations, Set Theory and General Thoughts

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Interview With a Set Theorist

Mirna Džamonja and Deborah Kant

Abstract The status of independent statements is the main problem in the philosophy of set theory. We address this problem by presenting the perspective of a practising set theorist. We thus give an authentic insight in the current state of thinking in set-theoretic practice, which is to a large extent determined by independence results. During several meetings, the second author asked the first author about the development of forcing, the use of new axioms and set-theoretic intuition on independence. Parts of these conversations are directly presented in this article. They are supplemented by important mathematical results as well as discussion sections. Finally, we present three hypotheses about set-theoretic practice: First, that most set theorists were surprised by the introduction of the forcing method. Second, that most set theorists think that forcing is a natural part of contemporary set theory. Third, that most set theorists prefer an answer to a problem with the help of a new axiom of lowest possible consistency strength, and that for most set theorists, a difference in consistency strength weighs much more than the difference between Forcing Axiom and Large Cardinal Axiom.

1.1 Introduction

The current situation in set theory is an exciting one. In the 1960s, set theory was challenged by the introduction of the forcing technique, in reaction to which some researchers might have turned their back on set theory, because it gave rise to a vast range of independence results. Today, the independence results constitute a large part of set-theoretic research. But how do set theorists think about them? How do mathematicians think about provably independent statements?

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An answer to this question can be attempted through a detailed description of the current set-theoretic practice through the eyes of set-theorists themselves, which is an enterprise to be realised. The present article provides a step towards that program by giving a description of some important aspects of set-theoretic practice, formulated and observed from a joint mathematical and philosophical perspective. During several meetings in Paris,[1] the second author (PhD candidate in philosophy of set theory) has been talking to the first author (a logician specialising in set theory and a professor of mathematics) in order to gain insights into the current situation in set theory, and to understand how set theorists think about their work and their subject matter. Parts of these conversations are directly presented in this article. They are supplemented by descriptive paragraphs of related (mathematical) facts as well as comments and discussion sections.

The article is structured as follows. First, we argue for the relevance of this article and our method. The second section contains facts of set theory and logic that will be relevant in the following sections. In the third section, we present some important forcing results, which includes mathematical details but is self-contained. We also added many (historical and mathematical) references. We then elaborate on classical, philosophical concepts that can be found in set-theoretic practice, for instance platonism. The fifth section presents some general observations about independence: Set theorists have developed some intuition which problems might turn out independent and which ones might be solvable in ZFC, and they can organise and differ between set-theoretic areas in this respect. Finally, we present three hypotheses about set-theoretic practice: First, that most set theorists were surprised by the introduction of the forcing method. Second, that most set theorists think that forcing is a natural part of contemporary set theory. Third, that most set theorists prefer an answer to a problem with the help of a new axiom of lowest possible consistency strength, and that for most set theorists, a difference in consistency strength weighs much more than the difference between Forcing Axiom and Large Cardinal Axiom.

The intended audience of this article includes both set theorists and philosophers of mathematics. Set theorists can skip Sect. 1.3, where some central mathematical concepts are introduced. For philosophers of mathematics who do not focus on set theory in their work, this section is intended to facilitate an understanding of the following text. They can also scan Sect. 1.4 without a loss of understanding for the subsequent sections. The aim of Sect. 1.4 is to show a variety of applications of the forcing method, and to support the view that forcing is an integral part of set-theoretic research.

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1.2 Methodological Background

We are interested in describing and analysing what set theorists do. We adopt Rouse's view that "a central concern of both the philosophy and the sociology of science [is] to make sense of the various activities that constitute scientific inquiry",[2] and apply it to set theory with a special focus on independence. In order to take a first step in the right direction, we present one specific view in current set-theoretic practice: the first author's view on her discipline. We cannot generalise this view to a description of set-theoretic practice, because other set theorists might have different views. Thus, we see the present article as an attempt to start a discussion on set-theoretic practice, which should lead to a more general, rigorous analysis.

1.2.1 How to Describe and Analyse Set-Theoretic Practice?

This question certainly deserves much more attention than we can devote to it in this article.[3] We briefly state our main points.

Our methods consist of a sociological method combined with philosophical considerations. On the sociological side, we can choose between two main methods: surveys and interviews. Surveys have the advantage of an easy evaluation of the information. They are suitable to check clearly formulated hypotheses. But there are two problems concerning hypotheses on set-theoretic practice: A missing theory and the question of a suitable language.

The study of set-theoretic practice is a rather new research area. In the literature, one can find specific views about set-theoretic practice,[4] or investigations of specific parts of set-theoretic practice,[5] but there is no analysis of current set-theoretic practice in general. This means that we do not have a theory at hand which could be tested in a survey. But we can use interviews to find reasonable hypotheses about set-theoretic practice.

Secondly, not only the theory has to be developed but also a suitable language for the communication between set theorists and philosophers has to be found. There are sometimes huge differences between the language which philosophers use and the language which set theorists use to talk and think about set theory. Therefore, communication between both disciplines can be difficult. There is a greater risk of a misunderstanding in a survey formulated by a philosopher and answered by a mathematician than in a question asked in an interview in which the possibility to immediately clear up misunderstandings is given. Hence, based on these aims of

[2] Rouse (1995, p. 2).

[3] But it will be considered more attentively in the PhD project of the second author.

[4] Maddy (2011) and Hamkins (2012).

[5] Rittberg (2015).

finding reasonable hypotheses and of a successful communication, we decided to conduct interviews.

On the philosophical side, a mathematical perspective is brought together with philosophical conceptions of mathematics. Furthermore, a mathematical perspective is communicated to philosophers, i.e. presented in a way that makes it comprehensible to other philosophers. The elaboration on our methodology in this section is part of the philosophical side of the methodology.

1.2.2 Why Describe and Analyse Set-Theoretic Practice?

We argue here that set-theoretic knowledge is not completely captured by gathering all theorems, lemmas, definitions, and the mathematical motivations and explications that mathematicians give to present their research. For example, when set theorists agree that cardinal invariants are mostly independent of each other, then we argue that this judgement, which is based on experience, is part of set-theoretic knowledge as well. Judgements of this kind are ubiquitous in set-theoretic practice. We extend our focus to set-theoretic practice in general. The notion of set-theoretic practice is taken here very generously as including all mathematical activities performed by set theorists as well as their thoughts and beliefs about mathematics (where the latter also includes definitions and theorems because we assume that set theorists believe what is an established definition and has been proven). In order to learn more about set-theoretic knowledge, we think that it is valuable to present the practices of the discipline, the similarities and differences between the views held by set theorists, and to formulate general ideas about the current situation in set theory.

The reasons for this are at least three, as we describe.

Visions Reflecting on the practices, the historical developments, the importance or role of specific objects and methods etc. gives rise to explicitly formulated visions for future set-theoretic research. What do set theorists wish to find out in the next ten years?[6]

Availability Set-theoretic knowledge is not easily accessible to other mathematical areas, the sciences in general and the philosophy of mathematics. With important exceptions, such as the philosopher Alain Badiou[7] or the musician François Nicolas[8] and of course many mathematicians and philosophers of mathematics, the independence phenomenon is not sufficiently known outside

[6]Large research programs have already been described by some set theorists. Consider, for example, the research programs by Hugh W. Woodin or Sy-David Friedman. For Woodin's program *see* Woodin (2017), and for Friedman's Hyperuniverse Program, *see* Arrigoni and Friedman (2013). Not everybody describes his/her program so specifically and we wish to discover more about these unspecified programs.

[7]Badiou (2005) bases ontology on set-theoretic axioms and also considers forcing.

[8]www.entretemps.asso.fr/Nicolas/

of the community directly studying set theory and logic. The formulation of set-theoretic ideas in a general and simple language could make set-theoretic knowledge available to other researchers who are interested in set theory.

Reasonable premises in philosophy In the philosophy of set theory, there is a debate on the independence problem and new axioms. Typical questions are *Is every set-theoretic statement true or false?* or *Which criteria can justify the acceptance of a new axiom?*[9] In some important philosophical approaches, set-theoretic practice plays a major role. For example, in Maddy's view, philosophical questions can only be answered adequately when considering in detail what set theorists are doing.[10] Hamkins' multiverse view is also strongly motivated by the current situation in set-theoretic research.[11] Therefore, some existing philosophical ideas require supplementation by an analysis of set-theoretic practice,[12] and such an analysis seems in general a promising starting point for future philosophical research.

1.3 Preliminary Facts

We summarise basic facts of logic and set theory such as the twofold use of the concept of set, the incompleteness of an axiomatic theory, independence proofs, the Continuum Hypothesis, forcing, and some new axioms. This section is intended to facilitate the understanding of the following interview parts, and it contains the necessary background for philosophical questions on set theory.

Set theory is the study of sets, and sets are determined by their elements. We can take the union of two sets, we can take their intersection, we can form ordered pairs and sequences. And we can consider infinite sets, like the set of all natural numbers or the set of all real numbers. Numbers can themselves be interpreted as certain sets,[13] and so can functions and many other mathematical objects. But also a sentence of a formal language can be interpreted as a certain set: every symbol of the formal language is interpreted as a set and then a sentence is just a finite sequence of these sets.[14] Therefore, also a formal theory—a set of axioms and all formal sentences that can be derived from these axioms—can be interpreted as a certain set.

[9] *See* the article by Laura Fontanella in this volume.

[10] *See* for example Maddy (2007, 2011).

[11] Hamkins (2012).

[12] Thanks to Carolin Antos for emphasising this fact.

[13] $0 = \emptyset$, $n + 1 = n \cup \{n\}$, $\mathbb{N} = \{n : n < \omega\}$, $\mathbb{R} = \mathscr{P}(\mathbb{N})$, and so on. (These equations should not be understood as the claim that numbers *are* sets.)

[14] For example, take the coding '$\exists$'=8, '$\forall$'=9, '$\neg$'=5, '('=0, ')'=1, 'v_0'=(2,0), 'v_1'=(2,1), then the statement '$\exists v_0 \forall v_1 \neg(v_1 \in v_0)$' can be coded as the sequence $\langle 8, (2, 0), 9, (2, 1), 5, 0, (2, 1), 4, (2, 0), 1\rangle$.

In set theory, the most commonly accepted formal theory is the Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC).

One could say that the concept of set in mathematics and in logic has an extremely wide scope of application. Set theorists study sets, and at the same time, they study the different models of ZFC (starting from the assumption that there is a model of ZFC). They can use their own sophisticated, set-theoretic methods on the models. (By contrast, when considering models of number theory, we cannot use number theory itself to manipulate the models but instead must resort to set theory.) Because of this twofold application of the concept of set, these two levels, which are sometimes distinguished as mathematics and metamathematics, are closely intertwined in today's set-theoretic practice.

The study of different models of ZFC is also the study of the independence phenomenon in set theory. The starting assumption to study a model is always that there exists a model of ZFC (by Gödel's Completeness Theorem for first order logic, this is equivalent to the assumption that ZFC is consistent).[15] This assumption itself is not part of ZFC (it could not be because of Gödel's second Incompleteness Theorem), but it is a natural assumption in practice. Thus, importantly, the set-theoretic method of building models is not a constructive method because of this starting assumption. If then, for a sentence φ, set theorists can build a model of the theory ZFC+φ, and they can also build a model of the theory ZFC+$\neg\varphi$, then φ is an independent sentence.

Let us give one example. In set theory, two sets have the same size if there is a one-to-one onto correspondence between them. So, the set of the natural numbers has the same size as the set of the even natural numbers, because $f : n \mapsto 2n$ is a one-to-one onto correspondence between them. All members of the first set can be completely paired up with the members of the second set. But if we take the real numbers as the second set, they cannot be completely paired up with the natural numbers (this is Cantor's Theorem). This gives rise to different sizes of infinite sets.

The size of sets is measured by cardinal numbers. 0, 1, 2, ... are cardinal numbers. For example, the empty set, $\emptyset$, has size 0, and the set that contains as its only element the set of natural numbers, $\{\mathbb{N}\}$, has size 1. The set of the natural numbers itself, $\mathbb{N}$, has size $\aleph_0$, which is the first infinite cardinal number (set theorists always start counting at 0). Of course, there are further cardinal numbers: $\aleph_1, \aleph_2, \ldots$ Now, set theorists have built models in which there are exactly $\aleph_1$ real numbers, and they have built models in which there are exactly $\aleph_2$ real numbers. Thus, the sentence "there are exactly $\aleph_1$ real numbers" (the famous Continuum Hypothesis (CH)) is an independent sentence. This can only be shown by building such models, and the most powerful technique to do this is forcing.

Forcing was introduced in 1963 by Paul Cohen,[16] who showed by this method that the Continuum Hypothesis is independent. The method was then adopted by the set theorists who have since then found (and continue to find) many independent

[15] Of course, any stronger assumption works as well, in particular any Large Cardinal Axiom.

[16] Cohen (1963, 1964).

statements. Different problems require different variations of the forcing method so that many kinds of forcing have been developed. This led to the formulation of Forcing Axioms. Such an axiom can be added to ZFC in order to facilitate the application of forcing. A Forcing Axiom for a certain kind of forcing states that any object that can be forced to exist by that kind of forcing already exists; it states that the forcing method already has been applied. These axioms are part of the new axioms in set theory.

It should be noted that the notion of a new axiom is not used by all set theorists. But in the philosophy of set theory, this notion includes all the axioms which are not part of the standard axiomatisation ZFC, but which are considered in set-theoretic practice.

In addition to the Forcing Axioms, there is another important class of new axioms—the Large Cardinal Axioms, which state that there exists a certain large cardinal. The smallest known large cardinal is an inaccessible cardinal. Other important large cardinals are measurable cardinals, Woodin cardinals, and supercompact cardinals (ordered by increasing strength). The existence of such large cardinals is not provable in ZFC (but for all we know, it might be that ZFC proves the non-existence of some of them!).

There are further statements, for instance determinacy statements, which are sometimes considered as new axioms, e.g. the Axiom of Determinacy (AD) which is consistent with ZF but contradicts the Axiom of Choice, and Projective Determinacy (PD) which is implied by the existence of infinitely many Woodin cardinals (Martin-Steel Theorem, 1985).

1.4 Some Important Forcing Results

This section briefly presents important steps in the development of the forcing technique. We first describe the first author's perspective on the moment of the introduction of forcing and the process of its adoption by set theorists. Second, we give an overview on subsequent inventions of different kinds of forcing, the conjectures they solved, and the formulation of Forcing Axioms.[17]

[17] Readers interested in the mathematical details of forcing are referred to Chow (2007) for an introduction, or Kunen (2011) for a classical presentation of forcing, or Shelah (2017) for a presentation of the forcing methods that are used today.

1.4.1 Cohen's Introduction of Forcing

Before the introduction of forcing by Paul Cohen in 1963,[18] there were no substantial independence results in set theory. It was already known that CH cannot be refuted since Gödel had constructed a model, L, of ZFC in which CH holds. L is an inner model of ZFC which is not obtained by the forcing method. M. Džamonja thinks that "many practising set theorists at that time were hoping, or were assuming, that CH or GCH[19] would be proven to be true." She points to a similar situation today: "Maybe just like now, we think that Large Cardinal Axioms are true in a sense, even though we cannot prove that they are." Today, Large Cardinal Axioms are an integral part of set-theoretic research. In comparison to other new axioms, they appear to be the most acceptable ones for set theorists. In general, most set theorists trust in the consistency of these axioms and do not believe that assuming them causes any harm. In other words, if they were forced to choose between the Large Cardinal Axioms and their negations, most set theorists would choose the former. Imagine now that this was the case with the Continuum Hypothesis before Cohen's result, which would mean that most set theorists did not expect that the negation of CH is a reasonable statement to consider.[20] This would explain why Cohen's proof was such a surprising result. In the following dialogue, we are speaking about this moment, the subsequent adoption of forcing by the set theorists, and, in particular, how forcing became a natural part of set theory.

D. Kant: *When Cohen's result was published, was it regarded as unnatural?*

M. Džamonja: *It was regarded as something very unnatural. There were many people who stopped working in set theory when they found out about this result. One of them was P. Erdös. He was a most prominent set theorist who had proved many interesting results, but he just didn't think that forcing was an interesting method or that it brings anything. Well, he has this famous statement that 'independence has raised its ugly head'. So, he didn't like it. He never learned the method. And, I think, in general, it was regarded of course as a big surprise. Cohen got a Fields medal for it. But it was very esoteric, and I think that even Cohen himself did not understand it the same way that we understand it now, after so many years, of course. Things become easier after many people had looked at them, and, yes, forcing was unnatural, totally unnatural. It is worth noting that one person, and one person only, had been entirely convinced that CH was going to be proven independent, and that person was Gödel. In spite of his*

[18] *See* Cohen (1963, 1964).

[19] General Continuum Hypothesis: For every ordinal α: $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

[20] Amazingly, this was not the case of Gödel, who shortly after discovering L and proving the relative consistency of GCH stated that he believes in the independence of GCH, see the beginning of the interview for this. But then, Gödel was considered a logician and a philosopher, not a mathematician, by the peers of the time.

own proof of the relative consistency of CH he wrote as early as 1947 that CH is most likely to be independent.[21]

D. Kant: *And, then, people quickly understood this technique and applied it?*

M. Džamonja: *Some people did, yes. It started in California and, well, there was Solovay, and there were also Dana Scott and many other people around Stanford and Berkeley, for example Ken Kunen and Bill Mitchell among the younger ones. Paul Cohen was in Stanford and Ken was a student in Stanford. So, I think, forcing was localized to the United States for a while. But, then, just a year or two later, it spread around to Israel. Yes, people did understand, but, I think, it was rather slow. I mean, the specialists understood perhaps quickly, but it was slow and it was not published. Cohen's book*[22] *took time to be published and it is not easy to learn the method from this book.*

D. Kant: *What would you say, when or with which results did forcing become more natural?*

M. Džamonja: *I think a subject generally becomes natural when people start writing and reading books about it. In this case, it was quite late. For example, Kunen's 'Set Theory'*[23] *came out in 1980, and that is really where people learned this from, from a book. Jech's book*[24] *also came out at that time. Before that, well, if you were at the right place at the right time you learned something about it. But it wasn't a well spread method. For example, I came from a country [Yugoslavia] in which there was a considerable amount of set theory, combinatorial set theory. But nobody was really doing forcing. I finished my undergraduate degree in 1984 and I wanted to write a thesis (we needed to write a thesis at the end of our undergraduate studies) on forcing. But I couldn't find an advisor for this in Sarajevo. So, I think, this tells you that people knowing this subject were rare from the Yugoslav perspective. Maybe Kurepa knew it, in Belgrade. I don't even know if he learned this method. I don't think he published any papers on it. And he was probably one of the greatest set theorists of the previous generation. In Hungary, I think, it took quite a while. It was maybe only when Lajos Soukup and Peter Komjath worked on this that it was seriously understood, and that was in the 80s. So, it took some time. In other countries of Eastern Europe, there were people like Bukovsky and Balcar who worked on this already in the 1960s, but it was politically difficult for them. Their work was practically unknown to others because of the Cold War. And, finally, in Russia, this method just didn't came through. Moti Gitik came from Russia to Israel thinking that he had discovered a new method: the method of forcing. He discovered it on his own, because he didn't have access to the research already known in the West. Probably, for someone of your generation, it's very difficult to imagine that time.*

D. Kant: *Yes.*

[21] Gödel (1947).

[22] Cohen (1966).

[23] Kunen (1980).

[24] Jech (2003).

M. Džamonja: *But literature was a really big problem. Because of the Cold War, and the other thing was the cost of journals. It was incredibly expensive. Except for top universities you wouldn't find in your university library journals that publish this kind of thing. So, it was really very restricted.*

We see that there was not so much resistance to adopt the forcing method. There was rather a kind of disappointment and resignation on the one hand, and interest and enthusiasm on the other hand. In addition, the circumstances of that time made the adoption of forcing a slow process. Nowadays, most set theorists know of the forcing method, although not everyone works with it.

The forcing technique has been developed for application to many problems.

1.4.2 Important Forcing Results

This section is intended to illustrate the set-theoretic research on independence. It contains more mathematical details than the other parts. We present important conjectures, theorems, and Forcing Axioms such as Easton's theorem, Suslin's Hypothesis, Martin's Axiom, the Borel Conjecture and the Proper Forcing Axiom, and we define many notions involved.

With Cohen's original method, one can prove that $2^{\aleph_0}$—which is the size of the continuum—can be any regular uncountable cardinal (and, even more generally, any cardinal of uncountable cofinality). A regular cardinal κ is one which cannot be obtained as a supremum of a sequence of cardinals of length less than κ. Conversely, a singular cardinal κ is one that can be reached in less than κ many steps. For example, the cardinal $\aleph_\omega$, which is greater than ω, can be reached in ω many steps: $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega$.[25] So, given an uncountable regular cardinal κ, one can force the statement '$2^{\aleph_0} = \kappa$'.

After Cohen presented this method, he did not pursue different applications of it. Instead, other people learned the forcing method, and applied it to solve many open problems. Robert Solovay was one of the most important pioneers in forcing from the early 1960s on. One should, of course, also mention Jack Silver and several other important set theorists from that time.

1.4.2.1 Easton Forcing

Solovay had many excellent students, including Matthew Foreman, Hugh W. Woodin, and William Easton. Easton introduced a new kind of forcing in his PhD thesis. Where Cohen only used a set of forcing conditions, Easton used a

[25] The correct notion to distinguish between regular and singular cardinals is the notion of cofinality. The cofinality of a cardinal κ is the length of the shortest sequence of ordinals less than κ which converges to κ.

proper class of forcing conditions.[26] This allowed him to show that the GCH can be violated almost arbitrarily. The only restrictions for cardinalities of 2^κ for regular κ are given by the requirements that $\kappa < \lambda$ implies that $2^\kappa \leq 2^\lambda$, and by König's Theorem:

König's Theorem (for cofinalities), 1905: *For every cardinal* κ*:* $\mathrm{cf}(2^\kappa) > \kappa$.[27]
Given this, Easton was able to present what is now called *Easton Forcing* and proved the following theorem.

ZFC+¬GCH, Easton's Theorem, 1970: *Let* F *be a non-decreasing function on the regular cardinals, such that* $\mathrm{cf}(F(\kappa)) > \kappa$ *for every regular* κ*. Then, by a cofinality and cardinality preserving forcing, we can obtain a model in which* $F(\kappa) = 2^\kappa$ *for every regular* κ.[28]

1.4.2.2 Suslin's Hypothesis, Iterated Forcing and Martin's Axiom

An old hypothesis about properties of the real line has become very important in set theory. It is based on a question asked by Michaił J. Suslin.[29]

Suslin's Hypothesis (SH), 1920: Every dense linear order in which there are at most countably many disjoint open intervals is isomorphic to the real line.
Given the definition of a Suslin Tree, the Suslin Hypothesis states that there is no Suslin Tree. This version is often used in practice.[30]

Suslin's Hypothesis is independent of ZFC. It was shown, independently by Stanley Tennenbaum in 1968 and Thomas Jech in 1967 (see Jech (2003) for historical remarks and references) that SH cannot be proved in ZFC. Likewise, Ronald Jensen proved that SH is false in L. He proved that the axiom V=L implies the $\diamondsuit$-principle[31]:

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[26] A proper class contains all sets that satisfy a given first order formula, but is itself not a set. (Thus, a proper class is defined by unrestricted comprehension over the universe of sets.)

[27] König (1905).

[28] Easton (1970, Theorem 1, pp.140-1). It is interesting to note that this is potentially class forcing. The Forcing Theorem of Cohen only applies to special cases of class forcing, so class forcing is less widely spread in applications. All of the forcing notions to follow will be set forcings.

[29] "(3) Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?" Sierpiński et al. (1920). Translation (by the authors): "Is a (linearly) ordered set without jumps nor gaps, such that every set of its non-overlapping intervals (containing more than one element) is at most countable, necessarily a linear continuum?"

[30] *See* Jech (2003, pp.114–116).

[31] 'Diamond-principle'.

$\diamondsuit$**-principle**, 1972: There exists a sequence of sets $\langle S_\alpha : \alpha < \omega_1 \rangle$ with $S_\alpha \subseteq \alpha$, such that for every $X \subseteq \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is a stationary subset of ω_1.[32] The sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ is called a $\diamondsuit$*-sequence*.[33]
Jensen proved that $\diamondsuit$ implies that there is a Suslin Tree:

ZFC + ¬SH: In ZFC+V=L, one can prove the existence of a Suslin Tree.[34]

Using a new technique, *iterated forcing*, Solovay and Tennenbaum showed that there is a forcing extension of ZFC+SH. This is the second part of the independence proof for SH. Iterated forcing consists in the transfinitely iterated application of single forcing notions. To make this work, one needs a *preservation theorem* which guarantees that the iteration of single forcings all satisfying a certain important property has still that important property. Here, that important property is the countable chain condition:

> Roughly spaking, it [the preservation theorem] says that the transfinite iteration of a sequence of Cohen extensions satisfies the countable chain condition (c.c.c.) if every stage satisfies c.c.c..[35]

To define this condition, we need to know that a *forcing notion* $(P, <)$ is a partial order, that two elements $p, q \in P$ are called *incompatible* iff there is no $r \in P$ such that $p < r$ and $q < r$, and that an *antichain* is a set $A \subseteq P$ such that all of its elements are pairwise incompatible.

Countable chain condition (c.c.c.): A forcing notion $(P, <)$ satisfies the *countable chain condition* if every antichain is at most countable.

The countable chain condition is an important property because c.c.c. forcing notions preserve cardinals and cofinalities. In general, when using an arbitrary forcing notion, things may happen that are undesired in the given context. Cardinals can be collapsed, cofinalities can be changed etc. preventing the respective statement from being forced.

Now, iterating certain c.c.c. forcing notions permitted Solovay and Tennenbaum to construct a forcing extension in which both ZFC+SH and a Forcing Axiom, called Martin's Axiom (after Donald A. Martin), hold:

Martin's Axiom (MA): For every c.c.c. forcing notion $(P, <)$ and every family of dense sets $\mathscr{D}$ such that $|\mathscr{D}| < 2^{\aleph_0}$ there exists a $\mathscr{D}$-generic subset $G \subseteq P$.[36]

[32] A subset of ω_1 is called *stationary* if it intersects all closed and unbounded subsets $C \subseteq \omega_1$, where C is *closed* if for every sequence $(a_n)_{n<\omega} \subseteq C$ the limit $\bigcup\{a_n : n < \omega\}$ is also an element of C, and C is *unbounded* if for every $a \in C$ there is a $b \in C$ such that $b > a$.

[33] Jech (2003, p.191).

[34] Jensen actually showed a more general version of which this theorem is one instance Jensen (1972, Theorem 6.2 and Lemma 6.5, pp.292–5).

[35] Solovay and Tennenbaum (1971) refer to theorem 6.3 on p.228.

[36] G is $\mathscr{D}$-generic means that $G \cap D \neq \emptyset$ for every $D \in \mathscr{D}$.

Given CH, Martin's Axiom is provable, because in this case $\mathscr{D}$ can only be countable, and for countably many dense sets there is always a generic set. Therefore, when using Martin's Axiom, it is often additionally assumed that $2^{\aleph_0} > \aleph_1$. In a footnote of their article, Solovay and Tennenbaum explain that they first worked without Martin's Axiom, but that Martin noticed a possible formulation in axiomatic terms.[37] However, they prefer using Martin's Axiom because it gives rise to a general proof scheme that is easier to apply than the repeated iteration of c.c.c. forcings.

> Most of the applications of iteration to date may be presented in the following manner. One shows that M [MA] (or possibly M + "$2^{\aleph_0} > \aleph_1$") implies a theorem T. Then the consistency proof for ZF + AC + M yields a consistency proof for ZF + AC + T.

To follow this approach, a relative consistency proof of Martin's Axiom is needed. Solovay and Tennenbaum gave such a proof as well. Once they had shown that Martin's Axiom is relatively consistent, it was enough to show that Martin's Axiom implies some theorem T. For, both results combined show that T is relatively consistent to ZFC. Solovay and Tennenbaum apply this scheme to SH and prove

ZFC + SH: "*Suppose ZF is consistent. Then so is ZF + AC + SH.*"[38]

1.4.2.3 Laver Forcing

Laver Forcing was developed to show the independence of the Borel Conjecture (named after Émile Borel).

Borel Conjecture, 1919: Every strong measure zero set is countable.[39]
Wacław Sierpiński proved in 1928 that the Borel Conjecture can be false.[40] To show that the Borel Conjecture can also be true, Richard Laver developed a forcing to add specific reals (*Laver reals*), which he then iterated.

ZFC + BorelConjecture, Laver Forcing, 1976: "*If ZFC is consistent, then so is ZFC+Borel's Conjecture.*"[41]

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[37] Solovay and Tennenbaum (1971, fn on p.232).

[38] Solovay and Tennenbaum (1971, Theorem 7.11 on p.242).

[39] Borel (1919). A *strong measure zero set* is a subset X of the reals such that for every sequence $\langle \varepsilon_n : n < \omega \rangle$ of positive real numbers there is a sequence $\langle I_n : n < \omega \rangle$ of intervals with $\text{length}(I_n) \leq \varepsilon_n$ such that $X \subseteq \bigcup \{I_n : n < \omega\}$.

[40] Sierpiński (1928).

[41] Laver (1976, Theorem on p. 152), *see* also Jech (2003, pp.564–8).

1.4.2.4 Proper Forcing and Proper Forcing Axiom

Proper Forcing was defined by Saharon Shelah.[42] He takes proper forcings to be well-behaved forcings, in particular because they behave nicely when they are iterated:

> When we iterate we are faced with the problem of obtaining for the iteration the good properties of the single steps of iterations. Usually, in our context, the worst possible vice of a forcing notion is that it collapses $\aleph_1$. The virtue of not collapsing $\aleph_1$ is not inherited by the iteration from its single components. As we saw, the virtue of the c.c.c. is inherited by the ... iteration from its components. However in many cases the c.c.c. is too strong a requirement. We shall look for a weaker requirement which is more naturally connected to the property of not collapsing $\aleph_1$, and which is inherited by suitable iterations.[43]

The weaker requirement Shelah is looking for is *properness*.

Proper Forcing: A forcing notion $(P, <)$ is *proper* if forcing with P preserves, for every uncountable cardinal λ, the stationary sets of $[\lambda]^\omega$.[44]
Many specific forcing notions can be shown to be proper. For instance, every c.c.c. forcing is proper.[45]

As for c.c.c. forcing, one also needs a preservation theorem to iterate proper forcing. This important result is due to Shelah.[46] Since iterating proper forcing works well, an axiom was formulated to simplify it.

Proper Forcing Axiom (PFA): For every proper forcing notion $(P, <)$ and every family of dense sets $\mathscr{D}$ such that $|\mathscr{D}| = \aleph_1$ there exists a $\mathscr{D}$-generic subset $G \subseteq P$.

Again, once the relative consistency of the Forcing Axiom is shown,[47] one can prove theorems as consequences of the Forcing Axiom without working through the details of the iterated forcing method. The consequences of a Forcing Axiom are then proven relatively consistent to ZFC. One important consequence of the Proper Forcing Axiom was proven by Boban Veličković and Stevo Todorčević:

In **ZFC + PFA**: *Assuming the Proper Forcing Axiom, one can prove* $2^{\aleph_0} = \aleph_2$.[48]
This means that in ZFC+PFA the Continuum Hypothesis is false.

Today, most forcings applied in practice are iterated forcings or the Forcing Axioms obtained by consistency proofs through iterated forcing. For example, a current research problem is to find properties of forcing notions which allow the existence of set-theoretic universes saturated for the generics for families of dense sets of size $\aleph_2$.

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[42] Shelah (1982).
[43] Shelah (2017, p.90).
[44] $[\lambda]^\omega$ is the set of all countable subsets of λ.
[45] Jech (2003, Lemma 31.2 on p. 601).
[46] *See* Shelah (2017, III. §3.).
[47] Assuming that there is a supercompact cardinal, one can prove that there is a model of ZFC+PFA.
[48] Jech (2003, Theorem 31.23 on p. 609).

1.5 Philosophical Thoughts in Set Theory

For someone who is interested in philosophical questions set theory is an exciting subject matter. A question that arises in set theory is for example the truth question: Are the independent sentences in set theory neither true nor false? What is truth in set theory if it does not coincide with provability? The truth question is part of traditional philosophy of mathematics. So is the question whether sets exist. If we assume that sets exist then we can easily give an account of truth: A sentence is true if and only if it is true in the universe of sets.

A closely related definition is the following: A set-theoretic sentence is true if and only if it is true in the intended model of set theory. This definition does not presuppose the existence of a universe of sets because it refers to the technical term of an intended model. Such a model may be given formalistically, i.e., as part of a formal theory. This definition is usually used when considering the formal number theory PA.[49] The standard model $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot, <)$ of number theory can be given in set-theoretic terms. So, it can be given in the formal theory ZFC. From a formalistic point of view, one could adopt the second definition (if one believes that there is an intended model) but not the first. And from the point of view of a platonist, the first and the second definition would correspond to each other—the intended model would be taken as the formal counterpart of the real universe of sets.

Such ideas about truth and existence are some of the philosophical thoughts that we find in set-theoretic practice. They are invoked, for example, as a justification of axioms. When the second author asked the first author about the meaning of the word "axiom", she had a clear answer: "For me, what it means is an obvious property of the intended universe," and she admitted that this "is a strong meaning because it implies the existence of an intended universe." This corresponds exactly to the above mentioned view of true set-theoretic sentences. But the word "axiom" is also used for the new axioms, which are either not generally accepted by set theorists or their acceptance is less immediate than the acceptance of the ZFC-axioms.

M. Džamonja: *When we think of axioms in the classical sense, we think of Euclid and his geometry, and the idea there is that the axioms are statements that are obvious. Obvious in the sense that we take some basic objects, which, I think, in Euclid's mind come from his intended application, which he takes as the only one, and then the axioms are certain statements that are obviously true about these basic objects. From these we build out further content. I think, the idea of axiomatic set theory was to do this but with mathematics in general. The basic objects are sets. So, certainly the Axiom of Pairing is obvious, even though, now with the Homotopy Type Theory, it's a complicated issue, but in the classical set theory, this type of axioms—Pairing, Union—are somehow clear. Well, some of the classical axioms are also less clear. Of course, an example is the notorious Axiom of Choice. It is not clear in what sense it is an axiom. And, in fact, maybe*

[49] PA stands for Peano Arithmetic which is the formal theory of the natural numbers.

that situation between ZF and ZF with Choice is in some sense similar to the situation between ZFC plus some Forcing Axiom or just ZFC because you might say that for some people the Axiom of Choice was not natural, they refused to work with it and they worked in ZF set theory. Or even, you can say that for those people working with the ZFC set theory it is also interesting to understand where one really needs the Axiom of Choice. So, to work somehow between ZF and ZFC. Certainly, then, one doesn't have to take an opinion of whether the Axiom of Choice is true philosophically or not, but can work in both ZF and ZFC and somehow take the neutral view that this is what I can do if I have the Axiom of Choice and this is what I can do otherwise.

So, if you take that view, then the Forcing Axioms are consistent extensions of ZFC. The first Forcing Axiom—the Martin Axiom—has the same consistency strength as ZFC. We do not need any extra Large Cardinal Axiom to prove its consistency. So, if we just concentrate on that one, if we look at ZFC vs. ZFC plus Martin Axiom, then we can say: 'Well, we don't have to take the view that Martin Axiom is true or not, but we could say that this is a possible axiom to add. Is this true or not, well, we don't know.' So, in that sense, it is reasonable to make this an axiom. Also, it was the first forcing-related statement that got in any way close to being, let us say, comprehensible to a large number of mathematicians and logicians. Once, we had the Martin Axiom, of course, the extensions of it started coming, like the Proper Forcing Axiom. They are extensions because, mathematically speaking, they are very similar to the Martin Axiom. Logically speaking they are not extensions because they require Large Cardinal Axioms, so they lose that property of equiconsistency with ZF that we had before. Considering ZFC or ZFC plus Proper Forcing Axiom for example, we have a much stronger consistency strength with PFA added. So, the two are not exactly at the same level. In the end, by extending the strength of these Forcing Axioms, we seem to get further and further from what an actual axiom might mean.

We have seen in these paragraphs that the word "axiom" can also be used as a rather technical term without any philosophical implications. This is also possible regarding the ZFC-axioms, but it is even more important to emphasise the possibility of such a *neutral view* regarding the new axioms. In this view, the use of the word "axiom" does not imply that this statement is in any way accepted as a statement itself. It is rather accepted as a reasonable, possible addition to ZFC. Thus, one can work in the corresponding theory to address mathematical questions without taking a stance on its acceptability.

But still, the answer that an axiom is either obviously true or just a possible statement to assume, does not seem to give the whole story. The role of Large Cardinal Axioms in set-theoretic practice could be a challenge to this view. Neither are they treated as on part with the axioms of ZFC, nor as mere opportune additions. For instance, every Large Cardinal Axiom (at least as strong as the existence of an inaccessible) implies the consistency of ZFC. This seems to support their acceptability.

D. Kant: *What do you say about the statement "ZFC is consistent"? Does it still play a role in set-theoretic practice?*

M. Džamonja: *Well, of course, in the beginning it was hoped that we will prove that ZFC is consistent. That was Hilbert's Program,*[50] *and then Gödel's results said: if we believe that ZFC is the basic theory, then we cannot, within that basic theory, prove that it is consistent. So, now, we have two choices: Either we accept just ZFC as our basic theory and then we have to take on faith that it is consistent, or we say 'Well, ok, I believe in large cardinals and then I get the consistency of ZFC for free', in the sense that, when we take the cumulative hierarchy and cut it at a large cardinal, we get a model of ZFC.*

This is actually the other side of your question of what is an axiom. If we have an axiom scheme that is supposed to be obvious, then it is supposed to be talking about the intended model. Now, there are people who do not believe in the intended model. I believe that there is a universe of sets, personally, this is my philosophical view. So, if there is such a thing, then the ZFC Axioms are the axioms of this universe. They are not the only axioms but they are the axioms that we accept. They describe this universe quite well, so, they have an intended model. They have other models as well. But, somehow, believing in the consistency comes back to thinking if there is this universe of sets or not. And, I think, this is now a philosophical question rather than a mathematical one.

D. Kant: *So, you would think that, among set theorists, the existence of a universe of sets is somehow subjective, and some believe in it and some do not?*

M. Džamonja: *Yes. For example, I think, Gödel was a very strong Platonist. Woodin confirms to be a very strong Platonist and he is searching for more complete axioms of set theory. I think Shelah also is a Platonist. But I know people, like Cummings for example, who told me some years ago that, for him, the question if there is an intended model or not is not at all interesting. What is interesting is that we get to do beautiful mathematics with these objects and if they exist or not is not that interesting. So, one can do the same mathematics independently of one's philosophical view. In fact, mathematicians in general, even set theorists who work in logic, do not always ask philosophical questions. Some do and some don't.*

In set theory, there are mathematicians who think about philosophical concepts such as platonism. However, it would be wrong to assume that every set theorist thinks about independence also in philosophical terms. Džamonja suggests that it is a matter of interest. There may be more people in set theory who are interested in philosophical questions than there are in other mathematical disciplines. Yet, not all set theorists are philosophically inclined.

In the above conversation, the possibility to believe in large cardinals is mentioned. This highlights an important difference between the Large Cardinal Axioms

[50] Hilbert wanted to prove the consistency of mathematics and focussed on axiomatisations of number theory. His program can be transferred to set theory, as set theory counts as a foundation of mathematics. Thus, of one would prove its consistency, Hilbert's aim would be resolved.

and the Forcing Axioms. We think that Forcing Axioms are often not seen as candidates for acceptance.[51] The function of a Forcing Axiom is not to capture a possible truth about the universe of sets, but rather to formalise a specific fruitful kind of forcing. The formulation of such an axiom makes the application of forcing easier because one does not have to build up the whole forcing machinery each time.

M. Džamonja also said explicitely that she believes in the existence of a universe of sets. She said that she believes the ZFC Axioms as well as the Large Cardinal Axioms. However, she made clear that these beliefs are relative to set theory. Since the universe of sets is an abstract reality, it is possible that it is not the unique reality for all of mathematics. This view is supported by research in Univalent Foundations and Homotopy Type Theory (HoTT). This mathematical field has its own concepts and methods, which differ significantly from other mathematical fields; it creates own content, and mathematics can be embedded in this theory. However, some set-theoretic principles do not generally hold there, e.g., the Axiom of Choice. With this in mind, M. Džamonja believes in the Axiom of Choice, but only restricted to set theory, not with respect to all of mathematics.[52] Both set theory and HoTT can serve as a foundation for much of mathematics. In a Platonist framework, both fields can have their own mathematical reality.

1.6 Set-Theoretic Intuition About Independence

Both the truth question and the existence question mentioned in the last section are part of classical philosophy of mathematics. However, these questions are not those which the second author seeks to answer by talking to set theorists. One can observe that, in set theory, one independence result is not similarly conceived of as another. Set theorists see differences in the value of the insights which they provide, or in the naturalness of independent statements. For example, some set theorists consider the axiom $V = L$ to be less natural than the existence of infinitely many Woodin cardinals, because $V = L$ is not compatible with the existence of many large cardinals. Woodin says: "[T]he axiom $V = L$ limits the large cardinal axioms which can hold and so the axiom is *false*."[53] The existence of infinitely many Woodin cardinals imply Projective Determinacy, which is an attractive statement for some set theorists, for example for Ralf Schindler: "The principle of projective determinacy, being independent from the standard axiom system of set theory, produces a fairly complete picture of the theory of 'definable' sets of reals."[54]

[51] Menachem Magidor certainly is an exception because he thinks that Forcing Axioms are natural axioms.

[52] For more on her view, see her article 'A New Foundational Crisis in Mathematics, is It Really Happening?' in this volume.

[53] His italics, Woodin (2010, p.504).

[54] www.math.uci.edu/node/20943 (06/05/2018)

In order to elaborate on such judgements, the authors talked about the question whether set theorists have an intuition about their subject matter which is based on their wide experience, but which is not necessarily backed up by proofs.

D. Kant: *I imagine that set theorists have gained a good intuition about what is provable in ZFC and what is independent. Would you say that you have a good intuition about this?*

M. Džamonja: *Yes, I think we do have a good intuition. Of course, not about everything, but about certain things, certain areas. I have a way that I see it. I think of the line of cardinal numbers. Certain areas of that line are well understood and we really have an intuition in that context, but other areas are murky.*

D. Kant: *Is there maybe something that you can say about these borders of ZFC? More concretely, is there something about these independent sentences that they have in common?*

M. Džamonja: *To start with, there are certain things that definitely cannot be independent because they are described by simple formulas and we have absoluteness theorems.*[55] *Things that are combinatorially close to them can likely be shown to be true or not true. So, descriptive set theory and things that go with it. We may find there some sort of mini-independence. They would be connected with certain classes of sets whose properties would be understood within a ZF bit which exhibits less absoluteness, such as analytic sets, or projective sets, etc.*
Sometimes, we can reflect independence to truth by restricting our attention to certain classes of sets. For example, suppose that we can use a Forcing Axiom to prove some statement about subsets of the reals in general. We can then hope to have the same statement hold about analytic sets without needing any additional axioms. Certain results that are obtained under PFA for general sets turn out to be true for analytic sets. For example, one can find this in the work of Todorcevic about gaps.[56] *There are analytic gaps, there are general gaps, there is the p-ideal dichotomy, and then there is this dichotomy applied to analytic objects. Or in the work of Solecki. So, that is one border of independence.*
Another border is, as I mentioned, the line of cardinal numbers. We know that at successors of regular cardinals we can do a lot by forcing, especially at $\aleph_1$. *We also know that at singular cardinals and their successors things are much more, let us say, resistant to forcing. This is so because we have pcf theory which shows that some things about singular cardinals are just true in ZFC, and, therefore, many statements that are implied by pcf theory are also just true.*[57] *So, there*

[55] For every Δ_0-sentence φ (a sentence with only bounded quantifiers) and every transitive standard model M of ZFC, $\varphi \leftrightarrow \ulcorner M \models \varphi \urcorner$ (in ZFC). Under the assumption that there is a transitive standard model of ZFC, this means that Δ_0-sentences cannot be independent. They are either provable or refutable. There are other well-known absoluteness theorems, such as the Shoenfield's Absoluteness.

[56] See for example Avilés and Todorcevic (2015, 2016).

[57] Shelah (2000) and Burke and Magidor (1990).

are these two distinct regions on the cardinal numbers line. There are successors of regular cardinals, which have some behaviour, and then there are singular cardinals and their successors, which have another one, and then, of course, there are large cardinals.[58] *So, we do have a good intuition when we start from a certain kind of cardinal in which direction to try to start working. And then we also have a good intuition about the kind of sentences as explained above. Combinatorial set theory is almost always about unrestricted sets. So, there we can expect to have a lot of independence.*

D. Kant: *About independent sentences in history: have there been some surprising or unexpected results? So, that, at first, the sentence was thought to be true, and then it turned out to be independent, or something like that? That really set theorists …*

M. Džamonja: *… were surprised?*

D. Kant: *Yes, were surprised about what came out?*

M. Džamonja: *Well, we have already said that the independence of CH came as a surprise to many mathematicians. But there is a recent example of just the opposite, when a statement was thought to be independent but at the end it turned out just to be true. I refer to a theorem by Malliaris and Shelah.*[59] *They proved a certain cardinal equality, that is, they proved that two cardinals,* $\mathfrak{p}$ *and* $\mathfrak{t}$, *which are cardinal invariants of the continuum, are actually just equal in ZFC. This was totally unexpected because there are many cardinal invariants known and they tend to be independent from each other. The independence of* $\mathfrak{p}$ *and* $\mathfrak{t}$ *was one of the last open questions and everybody expected they would behave like any other invariants, be independent—but they are not! The Malliaris-Shelah proof is also very ingenious, it mixes many different methods. That proof obtained an important prize in 2017, the Hausdorff medal that is given biannually by the European Set Theory Society for the most influential work published in the last five years.*

D. Kant: *But this does not happen very often?*

M. Džamonja: *No. That does not happen very often. Well, see, what I think is that, in mathematics, a huge percentage of results is proving something that is not so surprising. Everywhere in mathematics, including set theory, there are results that everybody suspects to hold, but if you want to be sure, you have to produce a proof. So, when somebody takes two new cardinal invariants and makes them independent of each other, that makes an ok PhD thesis but it does not make a huge surprise or a Hausdorff medal. Because we have seen such results very often. The opposite is surprising.*

This conversation should make clear that set theorists can say something more about the independence phenomenon than what they can prove. They can sometimes give probabilistic statements about the mathematical objects they work with. One

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[58] For a philosophical discussion of these regions see Džamonja and Panza (2018).

[59] Malliaris and Shelah (2013).

example was mentioned: Two cardinal invariants are often independent. Of course, such general ideas are based on experience and could turn out to be wrong—just imagine that there will be found many cardinal invariants such that pairs of them can often shown to be equal. However, such probabilistic, experience-based statements seem to play a very important role in set-theoretic research. In addition, it is an interesting philosophical question whether and, if so, how such general ideas can be seen as a part of set-theoretic knowledge.

For a first exploration on this question, we would argue that experience-based statements are part of set-theoretic knowledge. Given potential future experiences, they would have to be relativised to a time-frame in which they correspond to beliefs of most set theorists. Describing this part of set-theoretic knowledge is valuable because it can explain the development of the discipline. Imagine that some day, a new axiom is accepted. Then this would be possibly seen as a surprising event when only looking at the theorems. However, it could possibly be explained by looking at the experience-based, probabilistic part of set-theoretic knowledge. Furthermore, normative judgements by set theorists also play an important role in this context. For instance, judgements concerning the naturalness or attractiveness of particular axioms. These might also be explainable via the informal part of set-theoretic knowledge.

1.7 Conclusion

We started with a philosophical perspective on the set-theoretic independence phenomenon. This mathematical phenomenon raises many questions and can appropriately be called independence *problem*. In mathematical terms, on the other hand, it is not clear whether the independence phenomenon is a problem or a mere mathematical fact.

Putting together our mathematical and philosophical perspectives, we gave an insight in contemporary set theory. We focussed on forcing in order to illustrate to what extent independence results determine today's research. After that, we presented Džamonja's views on various topics, such as the introduction of forcing in set theory, the use of new axioms in practice (distinguishing between ZFC-axioms, Forcing Axioms, and Large Cardinal Axioms), the notion of the universe of sets, and surprising events. With this, we attempted to grasp what Tao calls the "solid intuition"[60] of an expert mathematician on her/his field of expertise. This solid intuition is grounded in many years of set-theoretic research, which seems to make it unavailable to non-set theorists. For, it is often the set-theoretic formalism and rigour which make it hard for philosophers and other mathematicians to acquire an understanding of the topics of set-theoretic research. However,

[60] Tao (2018).

> it is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the fine details, and the latter to correctly deal with the big picture.[61]

Given this, we can hardly hope to widely communicate the fine details of set-theoretic research. We can hope, however, to communicate a big picture in a comprehensible and correct way. It would be correct if it is consistent with set-theoretic practice. Such a big picture will include different perspectives of set theorists. They will differ on some aspects and agree on others.

To close this discussion, we want to leave the reader with a question and three hypotheses.

Question: What are the different aims/motivations for the uses of different axioms?

1. Hypothesis: Most set theorists were surprised by the introduction of the forcing method.
2. Hypothesis: Most set theorists think that forcing is a natural part of contemporary set theory.
3. Hypothesis: Most set theorists prefer an answer to a problem with the help of a new axiom of lowest possible consistency strength. And for most set theorists, a difference in consistency strength weighs much more than the difference between Forcing Axiom and Large Cardinal Axiom.

We distinguished a neutral view toward the use of Forcing Axioms on the one hand, and the use of accepted Large Cardinal Axioms on the other. This analysis can certainly be refined. The hypotheses are motivated because they correspond to Džamonja's view, which, we think, is representative for other set theorists as well. However, certain objections may be levelled. In contrast to the first hypothesis, one could also support a historical view according to which time had simply come for the forcing method to be introduced. The second hypothesis may be challenged by views of descriptive set theorists who rarely use forcing in their research. Finally, one might certainly find set theorists who would not agree with the third hypothesis (for example, for many years the school of the set theory of the reals did not accept the idea of large cardinals). Thus, we are not in the position to draw final conclusions. Rather, we encourage further research on set-theoretic practice which will bring further clarification.

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[61] ibid.

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Chapter 2
How to Choose New Axioms for Set Theory?

Laura Fontanella

Abstract We address the problem of the choice of new axioms for set theory. After discussing some classical views about the notion of axiom in mathematics, we present the most currently debated candidates for a new axiomatisation of set theory, including Large Cardinal axioms, Forcing Axioms and Projective Determinacy and we illustrate some of the main arguments presented in favour or against such principles.

2.1 Introduction

The development of axiomatic set theory originated from the need for a rigorous investigation of the basic principles at the foundations of mathematics. The classical theory of sets ZFC offers a rich framework, nevertheless many important mathematical problems (such as the famous continuum hypothesis) cannot be solved within this theory. Set theorists have been exploring new axioms that would allow one to answer such fundamental questions that are independent from ZFC. Research in this area has led to consider several candidates for a new axiomatization such as Large Cardinal Axioms, Forcing Axioms, Projective Determinacy and others. The legitimacy of these new axioms is, however, heavily debated and gave rise to extensive discussions around an intriguing philosophical problem: *what criteria should be satisfied by axioms?* What aspects would distinguish an axiom from a hypothesis, a conjecture and other mathematical statements? What is an axiom after all? The future of set theory very much depends on how we answer such questions. Self-evidence, intuitive appeal, fruitfulness are some of the many criteria that have been proposed. In the first part of this paper, we illustrate some classical views about the nature of axioms and the main difficulties associated with these positions. In the second part, we outline a survey of the most promising candidates for a new

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axiomatization for set theory and we discuss more specific arguments that were suggested in favor of these statements. In the end, we will not answer the question *How to chose new axioms for set theory?*, but we will discuss the main challenges associated with this problem. We assume basic knowledge of the theory ZFC.

2.2 Ordinary Mathematics

Before we start our analysis of the axioms of set theory and the discussion about what criteria can legitimate those axioms, we should address a quite radical view based on the belief that 'ordinary mathematics needs *much less than ZFC or ZF*'. This claim suggests that strong axioms such as the Axiom of Choice, or Infinity are not really needed for standard mathematical results, and certainly ordinary mathematics does not need new strong axioms such as Large cardinals axioms, Forcing Axioms etc. If so, then our goal of securing the axioms of ZFC and the new axioms would simply be irrelevant or a mere set theoretic concern (where set theory is not considered standard mathematics).

The issue with this view is to clarify what counts as 'ordinary mathematics'. In fact, the Axiom of Choice is heavily used in many fields such as algebra, general topology, measure theory and functional analysis. For instance, the Axiom of Choice is indispensable for the following claims and theorems (they are actually equivalent to the Axiom of Choice):

- For every equivalence relation there is a set of representatives.
- Every surjection has a right inverse.
- Every vector space has a basis.
- Krull's theorem: Every unital ring other than the trivial ring contains a maximal ideal.
- Tychonoff theorem.
- In the product topology, the closure of a product of subsets is equal to the product of the closures.
- Every connected graph has a spanning tree.

Other weaker consequences of the Axiom of Choice cannot be proven within ZF:

- Baire category theorem (which is equivalent to the Axiom of Dependent Choice).
- Hahn Banach theorem.
- Every Hilbert space has an orthonormal basis.
- The closed unit ball of the dual of a normed vector space over the reals has an extreme point.
- Every field has an algebraic closure.
- Stone's representation theorem for Boolean algebras.
- Nielse-Schreier theorem: every subgroup of a free group is free.
- Vitali theorem: there exists a set of reals which is not Lebesgues measurable.
- The existence of a set of reals which does not have the Baire property.
- The existence of a set of reals which does not have the perfect set property.
- Every set can be linearly ordered.

Thus, important applications of the Axiom of Choice can be found in many areas, hence rejecting the Axiom of Choice would come with a big price for the scope of 'ordinary mathematics'.

The analogous claim that ordinary mathematics does not need *more than ZFC* runs into a similar problem, as it is often the case that natural questions, that were raised in what one might consider a standard mathematical framework, turned out to be independent from ZFC, thus requiring stronger additional axioms to be settled. It is the case, for instance, for Whitehead problem in group theory: formulated in the 1950s, Whitehead problem was considered one of the most important open problems in algebra for many years, until S. Shelah showed in 1974 its undecidability in ZFC (see Shelah 1974); Whitehead conjecture is true if we accept the axiom of constructibility, namely that every set is constructible. A similar case is the Normal Moore Space Conjecture, a topological problem whose solution was eagerly sought for many years until strong large cardinal assumptions turned out to be indispensable for its solution. The reader is certainly familiar with the famous Fermat's conjecture recently demonstrated by Wiles who won the Abel prize for his outstanding result; what the reader might not be aware of, is that Wiles's proof relies on Grothendieck's universes whose existence requires large cardinals, namely strongly inaccessible cardinals (for more details, see McLarty 2010). It is generally believed that eventually we will be able to prove Fermat's conjecture in Peano Arithmetic (at the cost of a much more complicated organization of the proof), yet the only known proof today – more than 20 years after Wiles released it – uses Grothendieck universes in fact.

Independence results have always caused a certain embarrassment in the community of mathematicians. When a mathematical problem is proven to be independent from ZFC, suddenly it is labeled as 'just set theoretical' or 'vague' and no longer mathematical in the traditional sense. A precise definition of what 'ordinary mathematics' means should then take into account this attitude towards those problems which, at first, seem to emerge naturally as intrinsically relevant questions for mathematical research, then are dismissed after proven to require strong axioms. A simple move would be to claim that independent problems are legitimate mathematical questions that yet are 'unsolvable'. In this perspective, then, any attempt to answer such questions with stronger assumptions can only be seen as speculative. Surely, many mathematicians navigate these lines of thoughts. For instance, when Nykos (1980) proved in 1980 the consistency of the Normal Moore Space Conjecture from a strongly compact cardinal, he titled his paper 'A *provisional* solution to the Normal Moore Space Conjecture' (emphasis mine). However, if any result assuming large cardinals were just 'provisional', as Nykos' choice of words suggests, then Large Cardinals Axioms would be nothing more than mere *hypotheses*. Yet, despite the general skepticism towards the legitimacy of these principles, the mathematical community seems to acknowledge them a different status, a stronger role. In fact, we can point out that Wiles's proof of Fermat's conjecture was well accepted by the community of number theorists despite the fact that it relies on inaccessible cardinals. Imagine that his proof were assuming Riemann hypothesis instead, would his result even be published? The supporter of the view that ordinary mathematics can all be done in ZFC, or in a much weaker system than ZF, needs to clarify what should be the status of independent problems and of the additional assumptions needed for their solution.

2.3 Intrinsic Motivations

The word 'axiom' comes from the Greek $\alpha\xi\iota\omega\mu\alpha$ 'that which commends itself as evident'. To these days, most of the mathematicians would consider axioms to be a *self-evident* propositions requiring no formal demonstration to prove their truth, but received and assented as soon as mentioned. This is the ideal meaning of the word 'axiom'. The problem with this view is that what counts as obvious, self-evident, intuitive or inherently true is highly subjective.

> I can in no way agree to taking 'intuitively clear' as a criterion of truth in mathematics, for this criterion would mean the complete triumph of subjectivism and would lead to a break with the understanding of science as a form of social activity. (Markov 1962).

Moreover, the self-evidence criterion is quite restrictive. Not only the new axioms considered in contemporary set theory such as Large Cardinal axioms are far from self-evident (not even their strongest supporters claim they are self-evident), but even the axioms of ZFC are not strictly obvious. Certainly, the Axiom of Choice and the Axiom of Infinity were not immediately received and assented as soon as mentioned, on the contrary they were extensively debated and a mild skepticism still survives.

> The set theoretical axioms that sustain modern mathematics are self-evident in differing degrees. One of them – indeed, the most important of them, namely Cantor's axiom, the so-called axiom of infinity – has scarcely any claim to self-evidence at all. (Mayberry 2000, p. 10)

Problems occur even if we reformulate the self-evidence requirement and consider the following criterion that we may call '*intrinsic necessity*':

> An axioms must have some intuitive appeal, however the axiom may not be immediately obvious, but it should ultimately occur to us that what the axiom states is true and it could not be otherwise.

This reformulation may legitimate those controversial axioms, such as the Axiom of Choice, that were not immediately accepted but were eventually welcomed and employed. For instance, while the well-ordering principle mainly encountered reluctance, the equivalent statement 'the cartesian product of a collection of non-empty sets is non-empty' seems to be better accepted by the mathematical community as a fundamental truth. Unfortunately, the criterion of intrinsic necessity is still problematic. There is no strong reason for believing that what the Axiom of Choice states could not be otherwise. In fact, as proven by Banach and Tarski, the Axiom of Choice is actually paradoxical as it implies a quite counterintuitive statement (Banach-Tarski paradox): given a solid ball in a 3-dimensional space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball. Thus the Axiom of Choice challenges basic geometric intuition, leaving a shadow on its alleged intrinsic necessity.

Maddy's analysis in Maddy (1988) shows that even the less controversial axioms of the theory ZF were not motivated by intrinsic reasons but rather practical ones. Consider for instance the Axiom of Foundation: first introduced in the form $A \notin A$ to block Russell's paradox, it is nowadays adopted in its stronger version "every set is well-founded". Reasons for reformulating the axiom in this way were not based on self-evidence, but originated from the belief that "no field of set theory or mathematics is in any general need of sets which are not well-founded" (Fraenkel et al. 1973, p. 88). Actually, non-well founded sets found applications in computer science to model non-terminating computational processes; there is a whole line of work in logic that deals with alternatives to the Axiom of Foundation, the work of Forti, Honsell (1983) and Aczel (1988) in this area was especially influential.

Today the Axiom of Foundation is better supported by the so-called '*Iterative conception*'. Roughly, this consists in the idea that sets must be obtained by an iterative process where at a first stage certain sets are secured 'immediately', then new sets can be obtained starting from the sets at the first level so to form a second level, and at each stage new sets can be defined from the ones introduced at the previous levels. Under the Axiom of Foundation, all sets can be obtained in this way, in fact the class of all sets V coincide with the *Von Neumann Universe* which is defined as follows. The level zero V_0 is the empty set, then the first level V_1 contains just the empty set, at each successor stage $\alpha + 1$, the level $V_{\alpha+1}$ is defined as the set of all subsets of V_α (in fact V_1 coincides with $\mathscr{P}(V_0)$), at limit stages λ, we let V_λ be the union of all V_α for $\alpha < \lambda$. The Von Neumann Universe is the class obtained from the union of all V_α's. The Axiom of Foundation is equivalent to V being equal to the Von Neumann Universe which is the main expression of the iterative conception just discussed. This is often considered to be an intrinsic justification for the Axiom of Foundation, yet it is not *obvious* that such an iterative process would exhaust all possible sets. On the other hand, the Von Neumann hierarchy certainly gives a very useful and elegant description of the class of all sets, thus the Axiom of Foundation has undoubtedly strong *practical* merits.

2.4 Extrinsic Motivations

We argued that intrinsic motivations such as self-evidence, intrinsic necessity etc. are subjective and restrictive. Those considerations led Maddy to claim that axioms are mainly supported by *extrinsic motivations*, namely by their success, or fruitfulness. The roots of this idea already appeared in Gödel (1947):

> Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success", that is, its fruitfulness in consequences and in particular in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. (Gödel 1947, p. 521)

It is important to stress that Gödel and Maddy mean different things with the term 'fruitful'. Let us discuss first Gödel's view. For Gödel, the fruitfulness of an axiom is measured in terms of '*verifiable consequences*'. This is a delicate notion that deserves several comments. How can we verify a mathematical statement? Is this verification the result of an empirical process? Gödel believed in some sort of perception of mathematical entities analogous to our perception of physical objects. Thus, in Gödel's view, the truth of certain mathematical statements imposes on us to the extent that our intuition provides us with some sort of perception of the mathematical objects involved; other statements are not given to us immediately by mathematical intuition, but they are supported by the 'evidence' of their consequences.

In more recent work, Magidor considers this mathematical verification to be directly connected to our empirical knowledge of the physical world:

> As far as verifiable consequences, I consider the fact that these axioms [large cardinals] provide new Π^1_0 sentences which so far were not refuted. In some sense we can consider these Π^1_0 sentences as physical facts about the world that so far are confirmed by the experience. (Magidor 2012)

Whatever meaning we accord to the expressions 'mathematical verification' and 'mathematical evidence', we should note that, as for natural sciences, a plurality of verifiable consequences cannot secure the theory with certainty, since even inconsistent mathematical theories can, at first, appear to have many verifiable consequences. Thus, we can only say of a given axiom or theory that it was not refuted *so far*. In other words, paraphrasing Popper, mathematical theories are not strictly speaking verifiable, they are only *falsifiable*; but this is not different from physics, chemistry or other sciences.

A more challenging remark is that mutually incompatible theories can all be 'successful'. Consider, for instance, the theory ZFC plus the axiom of measurable cardinals versus ZF + V = L (more details about these axioms will be provided in the second part of this paper). These two theories are incompatible, yet none of their consequences was 'refuted' so far. In Gödel's view the axiom V = L should be rejected because it implies the continuum hypothesis which he believed was 'false' due to certain consequences that he considered 'highly implausible': for instance, CH implies the existence of a set of size $2^{\aleph_0}$ which has Lebesgue measure zero, but is not absolute zero. Yet, his feeling that such consequences would be implausible is not unanimously shared, thus we are left with an unclear notion of mathematical evidence which cannot guide us in the choice of one theory over another.

Maddy developed a wider conception of extrinsic justifications that she describes as based on *practical, inter-theoretic motivations*. In Maddy (2011), she explains the success of an axiom on the basis of the '*proper methods*' of the theory considered, namely the fruitfulness of an axioms is measured in terms of its effectiveness to achieve specific mathematical goals. So, for instance, Projective Determinacy came into considerations as the result of a broader research for new axioms that might settle certain problems in analysis and set theory that could not be solved within ZFC, accordingly certain Large Cardinals axioms are justified as they imply Projective Determinacy and settle other problems that are independent from ZFC.

Once again, distinct incompatible theories can be equally successful, even in this sense. For instance, if many strong Large Cardinals Axioms such as the axiom of measurable cardinals can be justified in this way, even the axiom of constructibility V = L can be viewed as an effective mean to achieve specific mathematical goals: V = L settles the continuum problem as it implies the Continuum Hypothesis and even GCH, it also implies the Axiom of Choice which can be used itself for proving classical fundamental theorems in mathematics, and it settles many other questions that are independent from ZF, for instance it implies the negation of the Suslin's hypothesis. Thus even this approach requires additional criteria for choosing a specific theory over the other. Maddy's suggestion is to appeal on the *maximality principle*. Roughly, this consists in the idea that we should prefer the theory that maximizes the concept of set. For instance, the concept of set underlying large cardinals seems to be wider than the one associated with the axioms of constructibility which is often ruled out as 'too restrictive'. Reference to this 'maximize rule' can be found for instance in Drake (1974), Moschovakis (1980) and Scott (1961). Nevertheless, the alleged restrictiveness of the axiom of constructibility was recently refuted by Hamkins (2014) who proved, roughly, that the axiom of constructibility is rich enough to allow one to talk about the concept of sets in the sense of large cardinals within a model of $V = L$. We will discuss this further in Sect. 2.5.

Finally, we can remark that the 'extrinsic approach' makes axioms depend on their consequences. This conflicts with the traditional view that considers axioms to be *the starting point for demonstration* from which, ideally, the truth of the other mathematical statements can be derived. Here, the situation is reversed: the consequences of an axiom legitimate the axiom, or they lead us to reject it when we have some 'counter-evidence' for such mathematical consequences. In this picture, then, any part of mathematics, including axioms, could be altered in the light of 'evidence'.

2.5 The Axiom of Constructibility

We now illustrate the main candidates for new axioms considered in contemporary set theory. We now disregard all the criticisms made so far of intrinsic and extrinsic motivations in general and we discuss more specific arguments that were suggested in favor or against these statements.

The oldest of these axioms is certainly the *Axiom of Constructibility* V = L that asserts that every set is constructible, namely every set belongs to Gödel's constructible universe L. L is inductively defined as follows:

- $L_0 = \emptyset$;
- $L_{\alpha+1}$ is the set of all subsets a of L_α that are definable with parameters in L_α (i.e. there is a formula $\varphi(x, a_1, \ldots, a_n)$ with parameters $a_i \in L_\alpha$ such that $a = \{x \in L_\alpha;\ L_\alpha \models \varphi(x, a_1, \ldots, a_n)\}$);

All these arguments seem to rest on mathematical platonism in an essential manner, as they appeal on some specific conception of 'the universe of sets' as uniform, inexhaustible, indescribable and so on. But even assuming a platonic point of view, what reasons do we have to believe that the universe of sets has such features? Some issues arise, for instance, with the claim of uniformity. In fact, there are properties that do hold at $\aleph_0$ and do not occur at higher cardinals. For instance, Ramsey's theorem establishes that for every $n, m < \aleph_0$ and for every coloring of the n-tuples of $\aleph_0$ into m colors, we can find a set $H \subseteq \aleph_0$ of size $\aleph_0$ such that all the n-tuples of H have the same color, this is called a *homogeneous set*; on the other hand, it can be proven that no uncountable cardinal can satisfy the same property: if we replace $\aleph_0$ with an uncountable κ, we get a statement that is provably false in ZFC.

Typically, large cardinals generalize properties of $\aleph_0$. For instance, the notions of *Ramsey cardinal, Erdös cardinal, weakly compact cardinals* and others can be defined as special generalizations of the theorem of Ramsey that we just mentioned; some limitations are necessary because as we said the direct generalization of Ramsey Theorem to an uncountable cardinal is provably false in ZFC. We consider, for example, the axiom of weakly compact cardinals which establishes the existence of an uncountable cardinal κ such that for every coloring of the pairs of ordinals of κ into less than κ many colors there is a homogeneous set of size κ. Once again, we stress the fact that generalizations are dangerous as they may lead to inconsistencies as in the case above.

The axiom of weakly compact cardinals can also be defined as a generalization of Compactness theorem to the infinitary language $\mathscr{L}_{\kappa,\kappa}$. Given two infinite cardinals κ, λ, we denote by $\mathscr{L}_{\kappa,\lambda}$ the infinitary language that roughly allows conjunctions and disjunctions of less than κ many formulas, and quantifications over less than λ many variables. Thus, for instance $\mathscr{L}_{\omega,\omega}$ corresponds to first order logic. An uncountable cardinal κ is weakly compact if, and only if, whenever we have a theory T in $\mathscr{L}_{\kappa,\kappa}$ with at most κ non logical symbols, if T is $< \kappa$-satisfiable (i.e. every family of less than κ many sentences of T is satisfiable), then T is satisfiable. If we remove the restriction to 'theories that have at most κ non-logical symbols, we have the notion of *strongly compact cardinal*. Other large cardinals axioms can be defined as generalizations of compactness theorem. Such generalizations imply interesting 'compactness results', namely given some structure, we assume that all its smaller substructures satisfy a certain property and we deduce that the whole structure satisfy the same property. For instance assuming a strongly compact cardinal κ it is possible to prove that every abelian group of size at least κ is free abelian whenever all its smaller subgroups are free abelian. The axiom of constructibility on the other hand is the 'cemetery of compactness properties': for instance, compactness for the freeness of abelian groups is actually false in $V = L$. The analysis of such compactness or incompactness results gives us no strong motivation to support one theory over the other, as there is no cogent reason to deem compactness more suitable than incompactness or the converse. Not even Uniformity helps us in this case, as ZFC proves both compactness and incompactness results: for example, König's lemma can be regarded as a compactness result,[2] but on the other hand its generalization to $\aleph_1$ is provably false in ZFC (there are Aronszajn trees).

We can see that the notion of weakly compact cardinal can be defined both as a combinatorial and a model-theoretic notion. The same occur for other large cardinals, in fact it is often the case that certain mathematical problems arising in completely different contexts and fields lead to the same large cardinal notions. This fact is sometimes considered to be an intrinsic motivation for large cardinals, but however remarkable this might seem, it is not clear how it can actually be considered as evidence for these axioms, rather than just a practical advantage.

The most powerful large cardinals axioms are the ones that can be defined as elementary embeddings of V into some inner model of ZFC. We discuss some of these notions in the next section.

2.7 Measurable Cardinals and Elementary Embeddings

In the history of large cardinals axioms the introduction of *measurable cardinals* was probably the most crucial step as it lead to the theory of elementary embeddings that are extremely useful in solving set theoretical problems and answering other mathematical questions. Let us discuss, then, these notions.

In 1902, Lebesgue formulated the measure problem: he asked whether there is a function that associates to every bounded set of reals a real number between 0 and 1 such that the function is not identically 0, it is translation invariant and countably additive. Motivated by this question, he introduced his famous Lebesgue measure (a function with these properties) and asked whether every bounded set of reals was Lebesgue measurable, namely whether his measure was defined over every bounded set of reals. Vitali soon found a counterexample under the Axiom of Choice, the problem was then reformulated by replacing the condition of translation invariance with 'every singleton must have measure 0', the minimal request for avoiding trivial solutions. The problem was still proven to be independent from ZF, in fact a counterexample can be built under CH. At this point Banach realized that the problem did not depend on the structure of $\mathbb{R}$, and it could be reformulated for a general set S : is there a function $\mu : \mathscr{P}(S) \to [0, 1]$ which is not identically 0, assigns to every singleton the value 0 and is countably additive? The solution of this problem comes down to the existence of certain large cardinal, the *real valued measurable cardinals*. A cardinal κ is real valued measurable if every set of size κ has a measure μ with the properties above which moreover is κ-additive, namely for every family $\{X_\alpha\}_\alpha$ of less than κ many sets, $\mu(\bigcup_\alpha X_\alpha) = \sum_\alpha \mu(X_\alpha)$. This is an example of how Maddy's approach to 'proper methods' works, namely real valued

[2] Given a tree of height ω whose levels are finite, if every finite subtree has a branch of the same length as the height subtree, then the whole tree also has a branch of the same length as the height of the tree.

measurable cardinals arose naturally as the solution to a specific mathematical problem.

Now, if we require that not only every set of size κ has a measure, but also the measure takes just two values 0 or 1, then we have *measurable* cardinals. In fact this notion has an extremely powerful characterization: κ is measurable if and only if one can define a non-trivial elementary embedding[3] $j : V \to M$ where M is a transitive class, such that κ is the least cardinal that is moved by j. By using this characterization, Scott was able to prove that if there is a measurable cardinal, then $V \neq L$. Thus, measurable cardinals, as well as any other stronger large cardinal, are incompatible with the axiom of constructibility.

Many powerful large cardinal notions can be defined in terms of elementary embeddings where we require the transitive class M to be 'closer' to V. The ultimate large cardinal notion expressible in terms of elementary embeddings is provably inconsistent with ZFC. This is the notion of *Reinhardt cardinal*, an uncountable cardinal κ for which there is a non trivial embedding j of V into itself where κ is the least cardinal which is moved by j.

Large cardinals axioms that establish the existence of elementary embeddings are more successfully justified by their fruitfulness, as they settle a number of questions that are independent from ZFC. The mort remarkable application of such cardinals is the theory of projective sets that under these cardinals gets a very elegant and exhaustive analysis. In fact, the existence of infinitely many Woodin cardinals implies that every projective set of reals is Lebesgue measurable, has the perfect set property and the Baire property, the so-called *regularity properties*. These considerations brings us to discuss Determinacy hypotheses, which is the object of the next section.

2.8 Determinacy Hypotheses

The study of regularity properties dates back to the earliest twentieth century from the work of the french analysts Borel, Baire and Lebesgue. Research in this area led to the development of an independent discipline, known as descriptive set theory. About 40 years later, it was shown that the open questions that descriptive set theorists were trying to solve (namely whether every set of real has the regularity properties above) could not be answered within ZFC (as we have seen for Lebesgue mesurability). In 1962 Mycielski and Steinhaus introduced the Axiom of Determinacy AD which was proven to solve such problems. AD is the assertion that every set of reals is *determined*, that means that for every set of reals A, one of the two players has a winning strategy in the following game of length ω. We regard A as a subset of ${}^{\omega}\omega$ (in set theory a real is an omega sequence of natural numbers),

[3]A function $j : V \to M$ is an elementary embedding if for every formula φ and parameters $a_1, \ldots, a_n$ one has $V \models \varphi(a_1, \ldots, a_n)$ if and only if $M \models \varphi(j(a_1), \ldots, j(a_n))$.

the two players I and II alternatively choose natural numbers $n_0, n_1, n_2, \ldots$ At the end of the game a sequence $\langle n_i; i \in \mathbb{N}\rangle$ is generated, player I wins if and only if the sequence belongs to A.

The Axiom of Determinacy implies that all sets of reals are Lebesgue measurable, have the perfect set property and the Baire property. Moreover, the statement that every set of reals has the perfect set property implies a weak form of the continuum hypothesis: every uncountable set of reals has the same cardinality as the full set of reals. On the other hand, AD implies the negation of the generalized continuum hypothesis. Despite its fruitfulness, AD was never seriously considered as a valid candidate new axiom for set theory as it contradicts the Axiom of Choice – here is a lucid example of the fact that often in mathematics the priority goes to the consequences rather than the axioms; but the consequences (here the Axiom of Choice) are themselves in need of justifications–. This led to investigate two distinct directions. The first approach was to assume AD in a quite natural subuniverse, namely $L(\mathbb{R})$, together with AC in the full universe V (L($\mathbb{R}$) is the smallest transitive inner model of ZF containing all the ordinals and the reals). The second approach was to consider a weakening of AD, called *Projective Determinacy*, PD. Projective Determinacy is the statement that every *projective* set of reals is determined. PD implies that every projective set of reals is Lebesgue measurable, has the perfect set property and the Baire property, and unlike AD, Projective Determinacy is not known to contradict the Axiom of Choice. Projective Determinacy follows from the existence of infinitely Woodin cardinals and this is the reason why this large cardinal assumption implies that every projective set of reals has the regularity properties above.

2.9 Ultimate L and Forcing Axioms

As we said, the Axiom of Constructibility and the Axiom of Determinacy both decide the continuum problem (the former implies GCH, the latter implies a weak form of CH, but it also implies the negation of the generalized continuum hypothesis). Large cardinals axioms, on the other hand, do not decide the size of the continuum. In this regard, a quite promising direction of research was considered which combine large cardinals with L and it may decide the size of the continuum, this approach is known as *V = Ultimate L*.

To understand this view, consider the intuition behind the Axiom of Constructibility: L is build up from a cumulative process where each stage is obtained from the previous one by a canonical operation, namely by taking the definable subsets of the previous stage; in this process only few ‘canonical’ sets are accepted at each stage. The idea behind V = Ultimate L is that, while we want large cardinals to exist in the universe of sets, we only want to include sets that are canonical or necessary after a fashion. Ultimate L, proposed by Woodin, is the alleged inner model for supercompact cardinals. Roughly this would be an L-like model where lives a supercompact cardinal. Such a model was not build yet and it is an open problem whether it can actually be found, but it can be proven that if the construction of the Ultimate L is successful, then it would contain also all the stronger large cardinals (i.e. stronger than supercompact cardinals). More importantly, V = Ultimate L would imply CH.

Magidor, however, expressed some doubts about this approach:

> It is very likely that the Ultimate L, like the old L, will satisfy many of the combinatorial principles like $\Diamond_{\omega_1}$. These principles are usually the reason that "L is the paradise of counter examples". They allow one to construct counter examples to many elegant conjectures. (The Suslin Hypothesis is a famous case). (Magidor 2012)

As for the Axiom of Constructibility, V = Ultimate L rests on the idea that sets are obtained through a cumulative process which is a way to allow only canonical sets of some sort, while other views rely on the opposite slogan that the concept of set should be as rich as possible. The most important example of such a liberal view is given by *Forcing axioms*. Forcing is the main tool for proving independence results in set theory. There are essentially two main approaches for building models of set theory and proving consistency results: one is through inner models which are obtained roughly by 'restricting' V into a subclass; the other is by using the Forcing technique where, conversely, V is expanded to a larger universe. Forcing axioms roughly establish that anything that can be forced by some 'nice' forcing notions (a forcing is simply a partially ordered set) is a set in the universe. For instance, the two most fruitful Forcing Axioms, PFA and MM, are the following statements.

The Proper Forcing Axiom PFA states that if P is a forcing notion that is proper and D is a collection of $\aleph_1$ many dense subsets of P, then there is a generic filter that meets all the dense sets in D.

Roughly, this says that anything that can be forced by a *proper forcing* is a set in the universe.

Martin's Maximum MM asserts that if P is a forcing notion that preserves stationary subsets of ω_1 and D is a collection of $\aleph_1$ many dense subsets of P, then there is a generic filter that meets all the dense sets in D.

Roughly, this says that anything that can be forced by a forcing that *preserves stationary subsets of* ω_1 is a set in the universe. We will not discuss these notions in the details, we should only point out that MM is the strongest possible version of a Forcing Axiom and it was proven to be consistent relative to the existence of a supercompact cardinal (this was suggested as another motivation for large cardinals axioms). Forcing Axioms settle many important questions that cannot be answered within ZFC, but more importantly they find remarkable applications in cardinal arithmetic. In fact, Foreman Magidor and Shelah proved in 1988 that Martin's Maximum settles the size of the continuum, it implies that the $2^{\aleph_0} = \aleph_2$. Later in 1992, Todorčević and Veličkovič showed that even the weaker axiom PFA implies that the size of the continuum is $\aleph_2$. Other remarkable applications of Forcing Axioms include the singular cardinals hypothesis (from PFA), the Axiom of Determinacy in L($\mathbb{R}$), the statement that any two $\aleph_1$-dense subsets of $\mathbb{R}$ are isomorphic (from PFA), every automorphism of the Boolean algebra $\mathcal{P}(\omega)/fin$ is trivial (from PFA), the $\aleph_2$-saturation of the ideal of non stationary sets on ω_1 (from MM), and the reflection of stationary subsets of κ for any regular cardinal $\kappa \geq \omega_2$ (from MM).

Chapter 3
Maddy On The Multiverse

Claudio Ternullo

Abstract Penelope Maddy has recently addressed the set-theoretic multiverse, and expressed reservations on its status and merits (Maddy, Set-theoretic foundations. In: Caicedo et al (eds) Foundations of mathematics. Essays in honor of W. Hugh Woodin's 60th birthday. Contemporary mathematics. American Mathematical Society, Providence, pp. 289–322, 2017). The purpose of the paper is to examine her concerns, by using the interpretative framework of set-theoretic naturalism. I first distinguish three main forms of 'multiversism', and then I proceed to analyse Maddy's concerns. Among other things, I take into account salient aspects of multiverse-related mathematics, in particular, research programmes in set theory for which the use of the multiverse seems to be crucial, and show how one may provide responses to Maddy's concerns based on a careful analysis of 'multiverse practice'.

3.1 The Problem

The development of set theory has progressively brought to the fore the problem of whether set theory should be interpreted as the theory of a *single* universe of sets, V, or whether it should be viewed as a theory about *multiple* structures (universes), that is, about a set-theoretic 'multiverse'.

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The universe/multiverse dichotomy is just an *ontological* (*semantic*) variant of the pluralism/non-pluralism dichotomy. The (proof-theoretic) pluralist believes that there is no unique, or preferred, theory T of sets, and that all theories express, to some extent, some properties of sets. For instance, the pluralist may believe that both ZFC+CH and ZFC+¬CH are equally valid and interesting theories of sets, insofar as they express alternative, but, in principle, equally acceptable, properties of sets. *Multiversism* is pluralism in its ontological/semantic form. It should be noted that one may be both a pluralist *and* a multiversist: that is, one may both believe that there are many, equally valid theories of sets and also that there are many, equally valid set-theoretic structures. Non-pluralism and, at the ontological level, *universism*, hold, respectively, that there is a single correct theory T of sets, or that there is a *unique* structure embodying all set-theoretic truths.[1]

The issue of which, between universism and multiversism, suits best set theory and its goals is crucial both for the philosophy of set theory and, more generally, for the philosophy of mathematics, as arguments in favour of multiversism may be taken to count as arguments in favour of the *absolute undecidability* of some set-theoretic statements, and we have evidence that multiversism is precisely construed in this way by some authors.[2]

The terms of the conceptions sketched above, and their relevance for the philosophy of mathematics, are rather uncontroversial to anyone. What, however, does seem contentious is whether the universe/multiverse dichotomy, and the possible adoption of a specific conception of the set-theoretic multiverse, is relevant to set-theorists and to their mathematical work. We know that set-theorists work with a plurality of models, such as models obtained through forcing, ultrafilter constructions, elementary embeddings, inner models like L, and many others. Now, how does the use of one (or more) collection(s) of set-theoretic models, that is, of the 'multiverse(s)' of a given theory T, contribute to their work? Are there any practical and foundational advantages in taking some collections of models, such as, say, countable transitive models or set-generic extensions of ZFC, as being especially relevant to set-theoretic work, and also as acting as a foundational framework for set theory?

This paper aims to provide answers to these questions, by using a specific, and especially authoritative, point of view, that of Maddian naturalism, and, in particular, by responding to the concerns that, in Maddy (2017), Maddy has expressed about the value and usefulness of a 'multiversist' approach.[3]

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[1] I am indebted to Koellner (2014) for this articulation of the pluralist/non-pluralist positions.

[2] This seems to be Väänänen's point of view in Väänänen (2014). Väänänen's main goal is to articulate a position which allows one to express the absolute undecidability of set-theoretic statements which are currently undecidable in several important theories (such as ZFC plus large cardinals). See also Sect. 3.2.1.

[3] It should be noted that the article mentioned is, in fact, an appraisal of *different* competing foundations of mathematics, also including set theory, and of the roles such competing foundations carry out. Only one specific section of the article is explicitly devoted to examining the prospects of the set-theoretic multiverse.

Over the years, Maddian naturalism has progressively, and coherently, come to the fore as one of the most influential positions in the philosophy of set theory, and in general, in the naturalist philosophy of mathematics. The position is generally associated to the following views:[4]

1. Metaphysical issues are irrelevant to the practical development of set theory, as well as to the justification of its internal techniques and results
2. The proper method of a naturalist philosophy of set theory consists in using rational methodologies attentive to intra-set-theoretic practice which altogether rule out extra-mathematical considerations
3. The justification and adoption of set-theoretic principles/axioms is also guided by methodologies of this sort

Especially points 1. and 2. in the summary above are crucial for set-theoretic naturalists: we ought not to evaluate mathematics, its goals and results, using extra-mathematical views or conceptions. The professed ideal of a 'second philosophy', that is of a philosophy of set theory which is especially attentive to *practice*, is the central tenet of the set-theoretic naturalist.[5]

Now, one further view distinctively associated to Maddy's naturalism is that our conception of the universe, V, is justified in light of set-theoretic practice, and in view of our set-theoretic purposes, insofar as set theory is pre-eminently guided by the strive to find a *unifying* account of mathematical phenomena. Therefore, if the 'unification' goal has priority over other goals, we ought to accept V as being the most suitable foundational framework for our set-theoretic investigations. This, in turn, sanctions the view that set theory essentially deals with proving facts about/establishing properties of V, a view which, as is clear, places Maddian naturalism in the universist camp.

In Maddy's works, the set-theoretic goal of unification has progressively taken the shape of the maxim 'unify'. Through fostering 'unify', set theory is thought to have been able to become the unique and far-reaching subject that it is today, and the maxim also serves as a spur to pursue further the search for solutions to the open set-theoretic problems. This point of view is articulated by Maddy as follows:

> If set-theorists were not motivated by a maxim of this sort, there would be no pressure to settle CH, to decide the questions of descriptive set theory, or to choose between alternative axiom candidates; it would be enough to consider a *multitude* [my italics] of alternative set theories. (Maddy 1997, p. 210)

[4]Throughout the paper, I shall use 'Maddian naturalism' and 'set-theoretic naturalism' (sometimes, just 'naturalism') interchangeably. Of course, there are many other ways to spell out naturalism in the philosophy of mathematics. An overview of all such positions is in Paseau (2016).

[5]Maddy herself has put forward and discussed the central aspects of set-theoretic naturalism as a form of 'second philosophy' in several works, starting with Maddy (1996). Second philosophy is further delineated in Maddy (2007), as well as in Maddy (2011).

As we know, the ‘independence phenomenon’ has introduced a rather different picture of the ontology of set theory, one which seems to be more compatible with a pluralist account of set-theoretic phenomena. This poses pressing questions for the Maddian naturalist described above: should she entirely disregard the issue of pluralism or actively engage with it? Moreover, since ‘unification’ concerns seem paramount in her account of set theory, on what alternative grounds may she adopt a pluralist picture? What will be of ‘unify’ within such a picture?

The paper also aims to explore these questions, with a view to providing arguments which may support the following two claims: the universe/multiverse dichotomy is relevant, in many ways, to the naturalist’s approach, and, secondly, the multiverse may be as acceptable as the universe, from a naturalist perspective, for the foundational purposes of set theory.

The structure of the paper is as follows. I first review several conceptions of the set-theoretic multiverse (Sect. 3.2), and provide a classification. I then proceed to summarise Maddy’s concerns (Sect. 3.3), which, overall, will take the shape of five main problems for the multiverse supporter. Finally, in the larger Sect. 3.4, I discuss aspects of the multiverse and of multiverse-related mathematics which seem to adequately respond to the issues raised in Sect. 3.3.

3.2 Multiverse Conceptions

First, we need to clarify what the set-theoretic multiverse consists in, and this may already be a daunting task. Even based on a minimal perusal of the existing literature, it is clear that there is no such thing as *one* mathematical conception of the set-theoretic multiverse, but rather *a bunch of* them, and, in addition, several, alternative research programmes which are variously connected to all such conceptions.

One main difficulty in addressing the set-theoretic multiverse, therefore, is precisely the absence of a shared framework wherein one may discuss results and methodologies. In what follows, I propose a classification of multiverse conceptions, which suits my specific goals. There is nothing compelling about the classification, nor is there any a priori need to classify multiverse conceptions, for that matter.[6]

[6]In Antos et al. (2015), the authors adopt a classification based on the realism/non-realism divide. Hamkins’ conception, for instance, counts as realist, whereas the Hyperuniverse Programme as non-realist. Väänänen proposes a different classification in Väänänen (2014), pp. 191–2: he divides conceptions into *countable* (Hyperuniverse Programme), *full* (Hamkins) and *set-generic* (Woodin, Steel).

3.2.1 Naive Multiversism

It is maybe noteworthy to mention that some form of multiverse thinking was already at work in the characterisation of the universe, since, historically, one could find the first description of a multiverse in Zermelo's characterisation of V. As is known, Zermelo proved the *(quasi-)categoricity* of his system of second-order set theory by showing that V_κ, where κ must be, at least, a strongly inaccessible cardinal, is, up to isomorphism, a model of the axioms ZFC_2 (that is, ZFC with second-order Separation and Replacement).[7] However, since there is an absolutely infinite collection of strongly inaccessible cardinals, one may generate V_κ's of increasing height, a 'tower' of universes, each the rank initial segment of the other. This could, in turn, be seen as a *height multiverse*, in which all universes have the same width (that is, no 'new' subsets may be added), but different heights ('new' ordinals, and ordinal-indexed stages, may be added).[8]

Leaving history aside, the most basic way to express a multiversist attitude is through taking note of the existence of many models of the axioms (of ZFC, for instance). As said at the beginning, we all know that set-theorists work with several kinds of models, through which they may, among other things, prove independence results. 'Naive multiversism' is just the idea that no single model $\mathcal{M}$ of a theory of sets T, should be viewed as 'special', as being *the* universe of sets, *the* collection of all sets.

Saharon Shelah, for instance, has compared models of the axioms to *individuals*, each, presumably, endowed with unique features, but all belonging to the *same* species.[9]

Naive multiversism could also be characterised in a different way, that is, as a conception which incorporates semantic pluralism in an ontologically *monist*

[7] See Zermelo (1930). An examination (and reprise) of Zermelo's proof is in Martin (2001).

[8] For Zermelo's height potentialism, see Linnebo (2013) and Ternullo and Friedman (2016). Of course, historically, Zermelo did not construe the ideas contained in Zermelo (1930) in the current multiversist terms. Väänänen suggests a different reason why Zermelo's characterisation of V ought not to be viewed as an instance of the multiverse. He notes that believing in the existence of an inaccessible κ in V means accepting the axiom: '$\exists\kappa$ inaccessible', which, although clearly independent from ZFC, is not indeterminate in the same sense as, say, CH is. According to Väänänen, the multiverse phenomenon takes place in the presence of statements which are indeterminate in the sense specified. In other terms, if V is V_κ, where κ is the least strongly inaccessible cardinal, then '$\exists\kappa$ inaccessible' is just false, and there is no 'parallel' V with inaccessibles. Cf. Väänänen (2014), p. 187. Finally, one could resist this interpretation of Zermelo's conception by asserting that the set-theoretic hierarchy is, in fact, fully *actual* in height and width. For a fuller examination of the actualism/potentialism dichotomy, see Antos et al. (2015) or Koellner (2009).

[9] Cf. Shelah (2003), p. 211. It is worth quoting the passage in full: 'My mental picture is that we have many possible set theories, all conforming to ZFC. I do not feel "a universe of ZFC" is like "the Sun", it is rather like "a human being" or "a human being of some fixed nationality"'.

Overall, also the set-generic multiverse exemplifies a practice-oriented attitude to the multiverse: in the same way as HP is guided by the goal of exploring maximality principles in (countable) models satisfying ZFC, Steel's set-generic multiverse aims to capture further set-theoretic truths in all set-generic universes which think ZFC+LCs.

3.2.3 Ontological Multiversism

The last strand of multiversism I describe is very different from all those described so far, and may legitimately be called 'ontological multiversism', that is, the view that the multiverse is a determinate *reality*, consisting of particular *entities*, the models of set theory. Hamkins' broad (radical) multiverse conception is such a form of multiversism.

Hamkins characterises this view as follows:

> ..the fundamental objects of study in set theory have become the models of set theory, and set-theorists move with agility from one model to another. [. . .] This abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for if one holds that there is a single absolute background concept of set, then one must explain or explain away as imaginary all of the alternative universes that set-theorists seem to have constructed. (Hamkins 2012, p. 418)

As to the issue of what universes there are in the multiverse, Hamkins says:

> The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP, and so on, perhaps even down to second-order number theory, as this is set-theoretic in a sense. At the same time, there is no reason to consider all universes in the multiverse equally, and we may simply be more interested in the parts of the multiverse consisting of universes satisfying very strong theories, such as ZFC plus large cardinals. (Hamkins 2012, p. 436)

Thus, Hamkins' multiverse is so broad as to include even non-well-founded models, and models of any collection of set- and class-theoretic axioms. Therefore, Hamkins' view is a form both of *proof-theoretic* and *model-theoretic* (*ontological*) pluralism I hinted at in the opening of the paper (p. 1). One peculiar feature of this conception is worth stressing straight away: according to its author, universes in the broad multiverse should be viewed as independent *existents*, just like more 'standard' mathematical entities (numbers, shapes, transfinite ordinals, etc.). The following quote illustrates Hamkins' full avowal of ontological realism (Platonism):

> The multiverse view is one of higher-order realism–Platonism about universes–and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. The multiverse view, therefore, does not reduce via proof to a brand of formalism. (Hamkins 2012, p. 417)

Let's take stock. There are at least three conceptions of the set-theoretic multiverse at hand, 'naive', 'instrumental' and 'ontological'. However, 'naive multiversism' does not seem to foster a sufficiently characterised notion of the set-theoretic multiverse. Therefore, in the rest of the paper, I will only be concerned with the other two forms of multiverse.

3.3 Maddy's Assessment of the Multiverse

I now proceed to summarise Maddy's concerns. For the sake of economy, I won't always quote Maddy's text in full, and, in some cases, I will just provide the references to the relevant pages in Maddy (2017).

Maddy notes that set-theoretic naturalists must acknowledge that set theory fulfils specific *foundational* roles, which are consistent with, and originate from, their picture of mathematics as being part of our *best* scientific theory of the world.[16] However, some of these roles are seen by her as spurious, others as appropriate.[17] Among the appropriate ones, Maddy lists 'Shared Standard', 'Generous Arena' and 'Metamathematical Corral'. 'Shared Standard' is the idea that set-theoretic proofs constitute the standard of 'proof in mathematics', whereas 'Metamathematical Corral' refers to the role played by set theory in allowing mathematicians to carry out metamathematical investigations, such as the search for consistency proofs.

'Generous Arena' is especially valuable to set-theoretic naturalists, insofar as it gives rise to and fully motivates the adoption of the meta-theoretic maxims 'unify' and 'maximize', which, in turn, justify the adoption of the axioms of set theory as our foundational theory, that is, a theory where all mathematical interactions among all mathematical objects take place.[18] Such a theory is ZFC (plus, possibly, LCs), interpreted as the theory of V.

Now, as a very general, overarching concern, one might legitimately doubt that the set-theoretic multiverse will fulfil the role of 'Generous Arena' equally well. The concern above may be summarised as follows:

Main Problem (Unification). *It is not clear whether and how the multiverse will fulfil the foundational role of a 'Generous Arena' (particularly, insofar as, at least*

[16]Some such foundational roles are also re-stated by Maddy in the paper published in the present volume.

[17]Rather unsurprisingly, among the spurious ones, Maddy mentions 'Metaphysical Insight', that is, the pretension that set theory provides us with an account of what mathematical objects *really* are, and 'Epistemic Source', *viz.* the idea that set theory provides us with an account of what mathematical knowledge is.

[18]The maxim 'maximize' was also introduced by Maddy in (1996). The application of the maxim to our conception of V implies that this has as many 'objects' as possible. It should be noted that, while, theoretically, maxims may be in tension with each other, they ought to be seen as having the same foundational (normative) content (cf. Maddy 1997, pp. 211–2). Among other things, this is shown by the fact that the iterative concept of set (which is generally taken to motivate V) is also naturally construed as being 'maximal' (for this, also see Wang 1974, Boolos 1971 or Gödel 1947). Cf. also Maddy (1996), in particular, pp. 507–12.

prima facie*, the multiverse provides us with a disconnected picture of set-theoretic phenomena).*

This concern can even become more general. Along with 'Generous Arena', there are further foundational roles expressed by set theory when 'standardly' construed as the theory of V, that the multiverse may not be able to fulfil. Maddy describes her further concerns as follows:

> The choice between a universe approach and a multiverse approach is justified to the extent that it facilitates our set-theoretic goals. The universe advocate finds good reasons for his view in the many jobs that it does so well, at which point the challenge is turned back to the multiverse advocate: given that we could work with inner models and forcing extensions from within the simple confines of V, as described by our best universe theory, what mathematical motivation is there to move to a more complex multiverse theory? (Maddy 2017, p. 316)

The troubles expressed in the quote above may be re-phrased as follows:

Problem 1 (Foundational Roles) *While we know what the universe can do for us, we do not know what jobs the multiverse can do for us, in particular whether it can successfully carry out all and the same (foundational) jobs that the universe does.*

As is clear, the Main Problem as well as Problem 1 strike both 'instrumental' and 'ontological multiversism'.

There is, however, one issue which relates specifically to 'ontological multiversism', which Maddy sees as particularly worrisome: metaphysics is so much involved in the characterisation of this position (one need only consider Hamkins' statements that his own view is one of 'higher-order realism, that is, Platonism about universes'), as to make it highly unsuitable to the set-theoretic naturalist. Therefore, we have:

Problem 2 (Metaphysics) *Multiversism heavily relies on metaphysics, in a way that the set-theoretic naturalist does not view as legitimate.*

One further foundational role of set theory is 'Conceptual Elucidation', a role that set theory has often held in replacing muddled and unclear mathematical concepts with sharper ones. As examples of 'Elucidation', Maddy mentions the formulation of the concept of 'continuity' in the nineteenth century, and 'the replacement of the imprecise notion of function with the set-theoretic version [...]'.[19] Now, can this foundational role also be carried out by the multiverse? The 'ontological multiverse' practitioner thinks that one of the roles associated to the multiverse is precisely that of exploring different 'concepts of set', by examining the *structures* which instantiate them.[20] But, for the Maddian naturalist, this is a sharply different

[19] Maddy (2017), p. 293.

[20] See Hamkins (2012), p. 417. Hamkins says: 'Often the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and in this article I shall simply identify a set concept with the model of set theory to which it gives rise. By adopting a particular concept of set, we in effect adopt that universe as our current mathematical universe; we jump inside and explore the nature of set theory offered by that universe.'

way of construing ‘Conceptual Elucidation’, a way which is strongly tied to the metaphysics evoked in Problem 2, as concepts, in this case, are also taken to be *objective* entities (p. 312, 315–6).

The naturalist multiversist, then, has to face up with one further problem, which, like Problem 2, fundamentally relates to ‘ontological multiversism’:

Problem 3 (Concepts) *‘Conceptual Elucidation’, construed as elucidation of* set concepts *instantiated by universes in the multiverse, should not be seen as a legitimate foundational role of set theory.*

There is one last, and fundamental, problem, with the multiverse, which strikes both ‘instrumental’ and ‘ontological’ multiversism. We have already seen how different multiverse conceptions are motivated by different research programmes in set theory. Now, the mathematics is, maybe, ok, but it is not clear, to the set-theoretic naturalist, that the multiverse construct is really essential to develop it, and, if yes, whether it can really act as a fully formal theory of sets. Maddy says:

> I’m in no position to evaluate the mathematics: my question is whether multiverse thinking is playing more than a heuristic role, whether there’s anything that couldn’t be carried out in our single official theory of sets. If not, then it’s not clear that these examples give us good reason to incur the added burden of devising and adopting an official multiverse theory as our preferred foundational framework. (Maddy 2017, p. 316)

We may re-phrase the concerns above as follows:

Problem 4 (Axioms) *It is presently not clear whether the multiverse is just a useful* heuristic *tool, or whether it is really* instrumental *for pursuing our set-theoretic investigations and, in particular, whether it will be able to replace the currently standard* axioms *of set theory.*

3.4 Addressing the Problems

3.4.1 Phenomenology of the Multiverse

I start with addressing Problems 2 (metaphysics) and 3 (Concepts). As we have seen, a recurring metaphysical claim made by ontological multiversists is that universes in the multiverse *really* exist. This claim implies, among other things, that ‘extensions’ of V exist. As we shall see, this is an especially problematic claim to make.

One further, equally problematic metaphysical claim is Hamkins’ assertion that there are different ‘concepts of set’ (as well as different concepts of ‘ordinal’, ‘cardinal’, ‘power-set’, and so on), construed, once more, as platonic objects instantiated by other platonic objects (set-theoretic structures).

3.4.1.1 Platonism and Existence

Hamkins' conception seems to re-state what has been known, for some time, as Full-Blooded Platonism (FBP), that is, that particular kind of Platonism which implies the existence of a *plenitude* of mathematical (set-theoretic) objects, as many as posited by all conceivable theories T of sets.[21] Let us now try to assess whether and how FBP really impacts on the mathematics of the multiverse. For instance, let us take into account what Hamkins calls the 'ontology of forcing'.

We know what the main problem with forcing is: set-theorists often use the notation $V[G]$ to refer to 'forcing extensions' of the universe, but, as is clear, V already has *all* sets, so it is not clear how it could possibly be extended. Clearly, FBP would have us view $V[G]$ as a fully meaningful (and existent) object, provided we can define it in a consistent way.

Surprisingly, though, Hamkins' account of forcing does not use any consciously FBP-inspired metaphysical principle, but rather what he calls a 'naturalist' account of forcing, based on purely mathematical facts. A full mathematical analysis of this is beyond the scope of this paper, but some details may be provided.

Hamkins proves that:

Theorem 1 (Hamkins) *Given the universe of sets V, it is possible to define an elementary embedding $j : V \to \bar{V}$, where $\bar{V}$ is a definable class* in V*, and a $\bar{V}$-generic filter G, such that $\bar{V} \subseteq \bar{V}[G]$, and $\bar{V}[G]$ is also a definable class* in V.

The crux of the theorem is that $\bar{V}$ and $\bar{V}[G]$ are definable classes *in* V and, thus, the naturalist account of forcing consists in showing that one can code extensions of V with subclasses of V itself. Already at this stage, the issue of the existence of such objects as $\bar{V}[G]$ becomes, in a sense, irrelevant. It is true that, given Theorem 1, one may use FBP to re-inforce the idea that such objects as $\bar{V}[G]$ *really* exist, but it is clear that the metaphysical content of FBP, is not instrumental, per se, for the proof of theorem.

Thus, the set-theoretic naturalist may simply want to take note of the methodology invoked by the theorem, but entirely disregard the metaphysical content attributed to it by FBP. In Hamkins' own words:

> This method of application, therefore, implements in effect the content of the multiverse view. That is, whether or not the forcing extensions of V actually exist, we are able to behave via the naturalist account of forcing entirely as if they do. In any set-theoretic context, whatever the current set-theoretic background universe V, one may at any time use forcing to jump to a universe $V[G]$ having a V-generic filter G, [...]. (Hamkins 2012, p. 425)

[21] FBP was introduced by Mark Balaguer in (1995). See also Balaguer (1998). Further details on Hamkins' use of FBP may also be found in Antos et al. (2015), pp. 2468–2470.

implied by V-logic statements φ asserting: 'there is an outer model of V which satisfies T', where T is an extension of ZFC.

So, it is only from the perspective of *Hyp(V)*, as defined in V-logic, that outer models of V can really be seen as existing.

In sum, the reference to 'reality' and 'illusion', thus, only serves to highlight more sharply the purposes and the extent of MP: the multiverse allows one to study inter-universe relationships by preserving our intuitive experience of this as a 'move' or a 'jump' from one universe to another.

3.4.2 Multiverse-Related Mathematics

I will now briefly present three case studies of 'multiverse-related' mathematics, which will help me illustrate that the multiverse may be able to fulfil many foundational jobs associated to set theory (which provides a response to Problem 1 (Foundational Jobs)), and also that it may not just be a useful heuristic, but rather a central construct in contemporary set theory, which partly responds to concerns expressed by Problem 4 (Axioms). I shall further address Problem 4 in Sect. 3.4.4.

3.4.2.1 Woodin's Set-Generic Multiverse and Ω-logic

I start with Woodin's results on CH in the set-generic multiverse. Woodin's guiding question was: is it possible to find an axiom which plays, for the structure $H(\omega_2)$, the same role as that played by PD for $H(\omega_1)$?[24] That is, is there any axiom which makes $H(\omega_2)$ 'well-behaved', as PD does with $H(\omega_1)$? Now, it turns out that it is relatively easy to force over properties of $H(\omega_2)$, which means that it is relatively easy to have different, mutually inconsistent pictures of $H(\omega_2)$. Therefore, Woodin identified the solution of the problem in identifying an axiom able to induce forcing-invariant properties of $H(\omega_2)$.

Now, it was known that many *forcing axioms* have this characteristic (that is, they are 'absoluteness axioms') and, moreover, that they imply the failure of CH. Therefore, Woodin's work was directed at identifying the appropriate forcing axiom which would make $H(\omega_2)$ well-behaved (and which, among other things, would also imply $\neg$CH), but the work carried out for this goal subsequently led to a parallel, equally fruitful, undertaking, that of defining a broader logical framework, wherein forcing invariance, in general, may be addressed. All this led Woodin to introduce a new logic, Ω-logic.

[24] $H(\kappa)$, for a cardinal κ, is the collection of all sets whose cardinality is hereditarily less than κ, that is, all sets whose elements and the elements of whose elements and so on, have cardinality less than κ.

Ω-logic is a logic in the full sense of the word, that is, a logical system which comes with its own definitions of semantic validity and logical consequence.[25] The semantics of Ω-logic is hinged on the use of Boolean-valued models $V^{\mathbb{B}}$ (where $\mathbb{B}$ is a complete Boolean algebra). The collection of all such models would, subsequently, become what Woodin defined the 'set-generic multiverse'.[26]

Now, it is important, for my purposes, to recall the definitions of validity and provability in Ω-logic. The definition of validity, with respect to a theory T, in Ω-logic (of $\models_\Omega$) is as follows:

Definition 1 (Ω-validity) $T \models_\Omega \phi$ if and only if, for all α ordinals, and all complete Boolean algebras $\mathbb{B}$, when $V_\alpha^{\mathbb{B}} \models T$, then $V_\alpha^{\mathbb{B}} \models \phi$.

The notion of provability is a lot more complex, as it uses *universally* Baire sets of reals, which cannot be addressed here.[27]

Definition 2 (Ω-provability) $T \vdash_\Omega \phi$ if and only if there exists an $A \subseteq \mathbb{R}$ universally Baire, such that $M \models \phi$, for every A-closed set M such that $M \models$ ZFC.

In turn, the Ω-conjecture is the conjecture that Ω-logic is complete (that is, that $\models_\Omega$ is equivalent to $\vdash_\Omega$).

As is known, in his (2001) Woodin, ultimately, focussed his attention on a specific forcing axiom, the $(\star)$ axiom, which allowed him to prove that, for all ϕ, ZFC+$(\star) \vdash_\Omega$ "$H(\omega_2) \models \phi''$ or ZFC+$(\star) \vdash_\Omega$ "$H(\omega_2) \models \neg\phi''$, precisely the kind of absoluteness result for $H(\omega_2)$ that Woodin was looking for.[28]

In the results mentioned, the foundational roles fulfilled by multiverse thinking are many. Woodin's 'Ω-logic solution' to CH looks very different from a 'standard', that is a solution consisting in showing that there is an axiom A which implies the truth or falsity of CH (something that Hamkins would subsequently define the 'dream solution' for CH).[29] The use of the Boolean-valued multiverse, or, more simply, of what would later become the set-generic multiverse, thus, stands out as an immensely successful way to elucidate statements of the complexity of CH, by

[25] Bagaria et al. (2006) is a comprehensive introduction to Ω-logic.

[26] Ω-logic makes its first appearance in Woodin (1999), and figures as a prominent tool in Woodin (2001). In those works, there is no direct reference to the set-generic multiverse, although the basic definitional ideas and concepts relating to it are already there.

[27] A rather accessible treatment of the provability relation in Ω-logic, and of universally Baire sets, is in Woodin (2011), p. 108.

[28] Moreover, Woodin was able to prove that:

Theorem 3 (Woodin) *ZFC*+$(\star) \vdash_\Omega$ "$H(\omega_2) \models \neg CH$".

It should be noted that the result requires the assumption of the existence of class-many Woodin cardinals, a particular strand of large cardinals having, as is known, far-reaching connections with Definable Determinacy Axioms, such as PD.

[29] See Hamkins (2012), p. 430.

showing, in particular, what such statements require in terms of 'solving resources', itself a way, in turn, to fulfil 'Conceptual Elucidation'.

Secondly, multiverse thinking leads to define a broader logical environment, that of Ω-logic, through which statements like CH (or ¬CH), in particular, proof-theoretic and semantic facts about them, can be represented. This, among other things, also implies a re-structuring of the notion of proof in set theory, something which should be viewed as being strongly connected with two further foundational roles, 'Shared Standard' and 'Metamathematical Corral'.

Finally, it should be noted that the kind of 'multiverse logic' inherent in the results mentioned is not only a specific 'tool' to be employed in representing facts about set-theoretic undecidability, but also a way to produce concrete mathematics, as shown by further work done on the Ω-conjecture.[30]

3.4.2.2 The Hyperuniverse Programme

As said in Sect. 3.2, the Hyperuniverse Programme (HP) has identified two main kinds of multiverse:

1. Zermelo's height multiverse[31]
2. The hyperuniverse $\mathbb{H}$, that is, the collection of all countable transitive models of ZFC

How did the programme get there? First came the proof that certain maximality principles have very important first-order consequences. For instance, take the IMH:

Definition 3 (IMH) For all ϕ, if ϕ is true in an inner model of an outer model of V, then ϕ is true in an inner model of V.

On the one hand, the IMH implies that there are no large cardinals in V (only in inner models of V), and that PD is false. On the other, refinements of IMH, like SIMH#, imply, among other things, that CH is false.[32]

Now, HP's maximality principles address extensions of V (in height and width). Therefore, one of the programme's main goals from the beginning has been to clarify what mathematical resources are needed in order to express principles which address extensions of V, like the IMH.

The answer consisted in adopting the multiversist position that we have mentioned above. In particular, in order to make maximality principles *mathematically* expressible, HP turned to taking into account:

[30] See, for instance, Viale (2016).

[31] See Footnote 8.

[32] Further details on all the different maximality principles explored by the HP may be found in Friedman (2016).

1. A partially *potentialist* view of *V*, whereby height extensions are admissible, but the width of the universe is fixed, which accounts for the introduction of the multiverse (1), or
2. A fully *potentialist* view of V, that is, a view whereby V is extendible in height and width, which accounts for the introduction of the multiverse (2)

By adopting (1), one may only state the IMH syntactically, as, by (1), outer models of *V* aren't really available, whereas, if one adopts (2), in particular, if one takes *V* to be countable, then one may have *real* 'thickenings' and, thus, *real* outer models satisfying the IMH. The latter choice leads to the introduction of $\mathbb{H}$.

As is clear, then, multiverse thinking is fully integrated in the mathematics of the HP, in the sense that it would be a lot more cumbersome to express maximality principles such as the IMH within a universist framework.

Moreover, we could say that, also within the HP, the multiverse helps one fulfil tasks which are associated to 'Conceptual Elucidation', such as elucidate what *V* is like, and also foster one's mathematical investigations on maximality principles and their consequences. Therefore, as in the case of Woodin's set-generic multiverse, the introduction of the hyperuniverse is not merely a way to represent facts about 'truth in V': it is a way to produce new mathematics, which subsequently leads to finding solutions to outstanding set-theoretic problems.

3.4.2.3 The Multiverse Case for $V = L$

One further, striking example of multiverse-related mathematics is Hamkins' multiversist construal of $V = L$. A very influential and widespread view concerning $V = L$ is that the axiom would be a sort of *minimality principle*, that is, a principle which implies that V is as small as possible.[33] Hamkins has attempted to challenge this point of view, by using mathematical facts which are deeply connected with the multiverse.

First, let us contrast the following two conceptions:

Conception 1 *There is an* absolute *background concept of set, and of other set-theoretic notions, such as set, ordinal, cardinal.*

Conception 2 *There is* no *absolute background concept of set and of other set-theoretic notions, such as set, ordinal, cardinal.*

[33] Of course, the fact that L is 'minimal' is simply a mathematical fact (insofar as L is the smallest inner model of V). As is widely known, the view that construes L as a 'minimality principle' has been expressed by Gödel in (1947), p. 478–9. In that work, Gödel explicitly contrasts $V = L$ to *maximum principles*. Maddy herself, as is known, has argued in favour of the claim that $V = L$ would be 'restrictive'. The full argument may be found in Maddy (1997), pp. 216–232.

The 'ontological multiverse' view, as has been said many times, is bound up with Conception 2, which, in turn, suggests the following facts: L may not have the same ordinals as V, insofar as the concept '(ordinal)^L' may be different from the concept '(ordinal)^V'. By adopting this approach, one may, then, proceed to establish further striking mathematical facts, all of which suggest that $V = L$ is not inherently restrictive. What follows is a summary of Hamkins' argument:

1. By Shoenfield absoluteness, statements such as 'T has a transitive model', which are Σ^1_2, are *absolute* between V and L. Therefore, even theories T which contradict $V = L$ (for instance, the theory ZFC+'∃ a measurable cardinal') have transitive models in V if and only if they have transitive models in L.
2. All reals in V can be coded in a model M of ZFC+V=L.
3. Any transitive countable model M can be 'continued' such that it may, ultimately, become a model of ZFC+V=L. Thus, using *forcing*, one may first collapse any model to the countable, and then use such model to make it satisfy $V = L$.[34]

The upshot of this is remarkable: a case for the non-limitative character of V=L can be successfully made within multiverse-related mathematics in a way which is entirely in accordance with the set-theoretic naturalist's desiderata.

Again, while talk of concepts might look suspect to the naturalist, I have already construed their use in an essentially non-metaphysical way through 'concept expansion' (in Sect. 3.4.1.2 which addressed Problem 2), and, based on this perspective, we may elucidate such notions as that of 'constructibility' and also completely revolutionise our mathematical perspective on such axioms as $V = L$.

3.4.3 Opposing the Argument from Priority and a New Unification

The argument I will be reviewing in this subsection re-states concerns expressed by the Main Problem (Unification) and Problem 1 (Foundational Jobs), and, therefore, here I will mostly be concerned with these two problems.

In particular, one may see the Main Problem as being introduced by an argument, which I will call 'argument from priority', which states that the drive towards a *unifying* account of set theory was a primordial goal of set-theoretic research, already inherent in the process of axiomatisation carried out by such pioneers as Zermelo, Fraenkel, von Neumann and Skolem. By virtue of this fact, this goal should still be viewed as a major goal of set-theoretic research.

[34]This leads Hamkins to even surmise that: 'For all we know, our entire current universe, large cardinals and all, is a countable transitive model inside a much larger model of $V = L$.' (Hamkins 2012, p. 436).

What Hamkins is setting out in the quote above is a fact already emerged in connection with Woodin's 'Ω-logic solution', that is, that finding a solution to CH implies taking into account the problem of what is needed, in terms of logical and mathematical resources, to solve it. However, this task cannot be executed, if one does not have a sufficiently broad collection of models (of a sufficiently strong theory) available, where the CH vs. $\neg$CH hypothesis may be tested, which is precisely what the multiverse provides us with.

Now, what I would like to highlight is the fact that the multiverse may, if viewed from such a foundational perspective, provide a different kind of unification, one which is needed to gain the sort of meta-theoretic 'knowledge' Hamkins is alluding to in the quotes above. In particular, one could suggest that the multiverse provides a different kind of 'Generous Arena' (a multiversist's 'Generous Arena'), one wherein all metamathematical interactions needed to provide us with knowledge about how to 'settle' problems of the same complexity, for instance, as CH, take place.

I argue that this may also be seen as a foundational role of set theory, which could be formulated as follows:

Multiversist's Generous Arena. *Set theory is also a systematic inquiry into the independence and unprovability phenomena, which provides us with knowledge about set-theoretic truth. In order to carry out such an inquiry, it is fundamental to have a unified metamathematical arena, where all relevant interactions take place*.

Of course, as already said, 'practically' one could still carry out such an inquiry within V. But should set-theorists agree that the Multiversist's Generous Arena is one further correct epistemological maxim for set theory to adopt, wouldn't the multiverse be the most natural candidate to fulfil it?

3.4.4 Relativism Reconsidered

Responses to the Main Problem (Unification), Problem 1 (Foundational Jobs) and, finally, Problem 4 (Axioms) are also provided by Steel's assessment of the goals of the multiverse in Steel (2014). We have already seen (in Sect. 3.2.2) that Steel's conception is an axiomatic version of the set-generic multiverse (MV).

Now, it is crucial, for Steel's purposes, to try to understand what MV really consists in. First, Steel shows that multiverse language is a sub-language of LST, that is, of the language of set theory. This is because a theorem proved by Woodin and Laver implies the following:

Theorem 4 (Woodin, Laver) *Given ϕ, $M^G \models \phi \leftrightarrow M \models t(\phi)$, where $t(\phi)$ is a formula saying: 'ϕ is true in some (all) multiverses obtained from M'.*[39]

[39]In Steel's notation, M^G is the set-generic multiverse containing all worlds satisfying MV.